Cut-free Proof Theory for Intuitionistic PDL

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Propositional dynamic logic (PDL) [4, 5] is a modal logic where modalities are indexed by an infinite number of *programs* π. Modal formulas of the form $[\pi]\phi$ are then interpreted as: *Every state reached after executing program π satisfies ϕ.* Programs π come equipped with *operations* generating new programs from pre-existing ones similar to that of Floyd-Hoare logic [12]:

- **Composition** ; if π_1 and π_2 are programs, then $\pi_1; \pi_2$ is the program that sequentially executed π_1 first, then π_2;
- **Choice** $\cup$ if π_1 and π_2 are programs, then $\pi_1 \cup \pi_2$ is the program that non-deterministically executes either π_1 or π_2;
- **Iteration** $*$ if π is a program, then π^* is the program which executes π a finite (possibly zero) number of times sequentially;
- **Test** ? if ϕ is a formula, then $\phi?$ is the program which tests whether ϕ holds, and continues execution if it does.

The formula $[\pi^*]\phi$ can be interpreted as $[\pi]^n\phi$ for all $n \in \omega$, i.e. *all states reachable from any number of executions (possibly zero) of π satisfies ϕ.* Equivalently, with fixed points, $[\pi^*]\phi$ can be seen to be equivalent to $\nu X(\phi \wedge [\pi]X)$. Due to this, proof theory of PDL has required an infinitary treatment to achieve a cut-free system, including: sequent calculi with ω-branching rules [9, 7]; infinite tableaux system [8]; non-wellfounded labelled sequent calculus [3].

Intuitionistic PDL (iPDL) has rarely been studied. A fragment was partially introduced by Leivant [9] and for the full fragment, a more extended semantical analysis was provided by Nishimura [10] who provides a sequent calculus with cut which is sound a complete with respect to the semantics.[1] We continue this line of work and explore a non-well founded sequent calculus for iPDL which is cut-free.

Axiomatisation. *Formulas* and *programs* of iPDL are generated by the grammar

$$\phi ::= p \in \mathsf{Prop} \mid \bot \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \to \phi) \mid [\pi]\phi \qquad \pi ::= \alpha \in \mathsf{BProg} \mid (\pi ; \pi) \mid (\pi \cup \pi) \mid \pi^* \mid \phi?$$

where Prop is a countable set of atomic propositions and BProg is a countable set of fixed *basic* programs. We write $\phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi)$.

The logic iPDL is an extension of intuitionistic propositional logic (IPL) with the following axioms:

$$\begin{array}{rcl@{\qquad}rcl}
\mathsf{k}_\pi & : & [\pi](\phi \to \psi) \to ([\pi]\phi \to [\pi]\psi) & [;] & : & [\pi_1;\pi_2]\phi \leftrightarrow [\pi_1][\pi_2]\phi \\
[\cup] & : & [\pi_1 \cup \pi_2]\phi \leftrightarrow ([\pi_1]\phi \wedge [\pi_2]\phi) & [*] & : & [\pi^*]\phi \leftrightarrow (\phi \wedge [\pi][\pi^*]\phi) \\
[?] & : & [\phi?]\psi \leftrightarrow (\phi \to \psi) & \mathsf{ind} & : & [\pi^*](\phi \to [\pi]\phi) \to (\phi \to [\pi^*]\phi)
\end{array}$$

with the modus ponens and necessitation inferences:

$$\mathsf{mp}\,\frac{\phi \quad \phi \to \psi}{\psi} \qquad \mathsf{nec}\,\frac{\phi}{[\pi]\phi}.$$

The axiom ind is sometimes called *Segerberg's axiom* and has similarities with the induction axiom of Peano/Heyting Arithmetic [14]. These axioms are also the standard ones used for classical PDL. To our knowledge, such an axiomatisation for iPDL is not present in the literature, however it can be shown to be sound and complete with respect to Nishimura's sequent calculus (with cut).

[1] Nishimura calls this logic 'constructive' however it follows the same intuitionistic 'i' discipline as Božić and Došen [2], and does not contain diamond modalities.

Semantics. Equivalently, iPDL can be defined semantically:

Definition 1. A *structure* is a tuple $\mathcal{M} = (W, \leq, \{R^\pi \mid \pi \in \mathsf{Prog}\}, V)$ where: $W \neq \emptyset$ is a set of *worlds*; $\leq$ is a *preorder* on W; each R^π is a *binary relation* on W; V is a map $V : W \to \mathcal{P}(\mathsf{Prop})$ with $V(w) \subseteq V(v)$ whenever $w \leq v$. Truth in a structure is defined inductively as follows

$$
\begin{array}{lcllll}
\mathcal{M}, w & \Vdash & p & \text{iff} & p \in V(w) \\
\mathcal{M}, w & \Vdash & \bot & & \text{never} \\
\mathcal{M}, w & \Vdash & \phi \wedge \psi & \text{iff} & \mathcal{M}, w \Vdash \phi \text{ and } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w & \Vdash & \phi \vee \psi & \text{iff} & \mathcal{M}, w \Vdash \phi \text{ or } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w & \Vdash & \phi \to \psi & \text{iff} & \text{for all } v \geq w, \text{ if } \mathcal{M}, v \Vdash \phi, \text{ then } \mathcal{M}, v \Vdash \psi \\
\mathcal{M}, w & \Vdash & [\pi]\phi & \text{iff} & \text{for all } v \text{ with } wR^\pi v,\ \mathcal{M}, v \Vdash \phi
\end{array}
$$

A *model* for iPDL is a structure $\mathcal{M} = (W, \leq, \{R^\pi \mid \pi \in \mathsf{Prog}\}, V)$ where:

- for each basic program α, R^α satisfies triangle confluence, i.e. if $w \leq vR^\alpha u$, then $wR^\alpha u$; as visualised in the following diagram

$$
\begin{tikzcd}
v \arrow[r, "R^\alpha"] & u \\
w \arrow[u, "\leq"] \arrow[ur, dashed, "R^\alpha"']
\end{tikzcd}
$$

- for programs π_1 and π_2, $R^{\pi_1;\pi_2} = R^{\pi_1} \circ R^{\pi_2}$ and $R^{\pi_1 \cup \pi_2} = R^{\pi_1} \cup R^{\pi_2}$;
- for program π, $R^{\pi^*} = \leq \cup (R^\pi)^+$ where $(R^\pi)^+$ is the transitive closure of R^π;
- for formula ϕ, $R^{\phi?} = \{(w, v) \mid w \leq v \text{ and } \mathcal{M}, v \Vdash \phi\}$.

We write $\Vdash \phi$ when ϕ is satisfied in all models as standard.

As some of these relations are defined on validity of a formula at a world, we separate the notion of structures and models to ensure it is well-defined. Triangle confluence is a requirement for logics using this intuitionistic discipline [11, 6, 2] to ensure monotonicity.

Proof theory. We employ the use of a *multi-succedent* sequent calculus. A *sequent* is an ordered pair $\Gamma \Rightarrow \Delta$ where Γ and Δ are *finite sets* of formulas. The *interpretation* of $\Gamma \Rightarrow \Delta$ is the formula $(\Gamma \Rightarrow \Delta)^I = \bigwedge \Gamma \to \bigvee \Delta$. The sequent calculus mLiPDL is given by the rules given in Fig. 1. The rules given are mostly standard but we note that $\to_r$, k_α and $[?]_r$ are non-invertible rules. We use a multi-succedent calculus to aid with the completeness proof which uses a proof-search tree.

A *preproof* is an infinite tree where nodes are sequents and edges are rules such that each node is the conclusion of a correct rule instance for which its children are premisses. Along an infinite branch of a preproof, a *trace* is a maximal path in the graph of ancestry, restricted to that branch.

A *proof* is a preproof where, along each infinite branch, there is a trace that eventually hits a formula $[\pi^*]\phi$ for which it is infinitely often principal and always follows the right premiss of the $[*]_r$ rule.

Example. The following is an example proof of an instance of the Segerberg axiom $[\alpha^*](\phi \to [\alpha]\phi) \to (\phi \to [\alpha^*]\phi)$.

$$
\to_r \dfrac{\to_r \dfrac{[*]_l \dfrac{\to_l \dfrac{\mathsf{id}\,\dfrac{}{\phi \to [\alpha]\phi, [\alpha][\alpha^*](\phi \to [\alpha]\phi), \phi \Rightarrow \phi, [\alpha^*]\phi} \qquad [*]_r \dfrac{\mathsf{id}\,\dfrac{}{[\alpha]\phi, [\alpha][\alpha^*](\phi \to [\alpha]\phi), \phi \Rightarrow \phi} \quad \mathsf{k}_\alpha \dfrac{[*]_l \dfrac{\vdots}{\phi, [\alpha][\alpha^*](\phi \to [\alpha]\phi) \Rightarrow [\alpha^*]\phi}\circ}{[\alpha]\phi, [\alpha][\alpha^*](\phi \to [\alpha]\phi), \phi \Rightarrow [\alpha][\alpha^*]\phi}}{[\alpha]\phi, [\alpha][\alpha^*](\phi \to [\alpha]\phi), \phi \Rightarrow [\alpha^*]\phi}}{\phi \to [\alpha]\phi, [\alpha][\alpha^*](\phi \to [\alpha]\phi), \phi \Rightarrow [\alpha^*]\phi}\circ}{[\alpha^*](\phi \to [\alpha]\phi), \phi \Rightarrow [\alpha^*]\phi}}{[\alpha^*](\phi \to [\alpha]\phi) \Rightarrow \phi \to [\alpha^*]\phi}}{\Rightarrow [\alpha^*](\phi \to [\alpha]\phi) \to (\phi \to [\alpha^*]\phi)}
$$

where $\circ$ indicates repeating the same branches infinitely often.

$$\mathsf{id}\ \frac{}{\Gamma, \phi \Rightarrow \phi, \Delta} \qquad \bot_l\ \frac{}{\Gamma, \bot \Rightarrow \Delta}$$

$$\wedge_l\ \frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta} \qquad \wedge_r\ \frac{\Gamma \Rightarrow \phi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \wedge \psi, \Delta}$$

$$\vee_l\ \frac{\Gamma, \phi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta} \qquad \vee_r\ \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}$$

$$\to_l\ \frac{\Gamma, \phi \to \psi \Rightarrow \phi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \to \psi \Rightarrow \Delta} \qquad \to_r\ \frac{\Gamma, \phi \Rightarrow \psi}{\Gamma \Rightarrow \phi \to \psi, \Delta}$$

$$\mathsf{k}_\alpha\ \frac{\Gamma \Rightarrow \phi}{\Pi, [\alpha]\Gamma \Rightarrow [\alpha]\phi, \Sigma}$$

$$[;]_l\ \frac{\Gamma, [\pi_1][\pi_2]\phi \Rightarrow \Delta}{\Gamma, [\pi_1;\pi_2]\phi \Rightarrow \Delta} \qquad [;]_r\ \frac{\Gamma \Rightarrow [\pi_1][\pi_2]\phi, \Delta}{\Gamma \Rightarrow [\pi_1;\pi_2]\phi, \Delta}$$

$$[\cup]_l\ \frac{\Gamma, [\pi_1]\phi, [\pi_2]\phi \Rightarrow \Delta}{\Gamma, [\pi_1 \cup \pi_2]\phi \Rightarrow \Delta} \qquad [\cup]_r\ \frac{\Gamma \Rightarrow [\pi_1]\phi, \Delta \quad \Gamma \Rightarrow [\pi_2]\phi, \Delta}{\Gamma \Rightarrow [\pi_1 \cup \pi_2]\phi, \Delta}$$

$$[*]_l\ \frac{\Gamma, \phi, [\pi][\pi^*]\phi \Rightarrow \Delta}{\Gamma, [\pi^*]\phi \Rightarrow \Delta} \qquad [*]_r\ \frac{\Gamma \Rightarrow \phi, \Delta \quad \Gamma \Rightarrow [\pi][\pi^*]\phi, \Delta}{\Gamma \Rightarrow [\pi^*]\phi, \Delta}$$

$$[?]_l\ \frac{\Gamma, \phi \to \psi \Rightarrow \phi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, [\phi?]\psi \Rightarrow \Delta} \qquad [?]_r\ \frac{\Gamma, \phi \Rightarrow \psi}{\Gamma \Rightarrow [\phi?]\psi, \Delta}$$

Figure 1: Rules of mLiPDL

We show the following:

Theorem 2. $\mathsf{mLiPDL} \vdash \Gamma \Rightarrow \Delta \iff \Vdash (\Gamma \Rightarrow \Delta)^I$

The method follows [13, 1]. The $\Rightarrow$ direction is proved by contradiction. The main idea is that if a formula of the form $[\pi^*]\phi$ is not valid, then there is a least $n \in \omega$ such that $[\pi]^n\phi$ is not valid. Following such a formula along a trace can only decrease this measure which leads to a contradiction.

The $\Leftarrow$ direction utilises prover-refuter games (of complexity Π^0_2) on the proof-search tree of the sequent which is *determined*; refuter winning leads to a countermodel construction; prover winning leads to a valid proof.

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