▶ SONIA MARIN AND PAARAS PADHIAR, *Nested sequents for quasi-transitive modal logics.*
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The modal logic of transitive frames (such that, if xRy and yRz, then xRz) is known to be $\mathsf{K4}$, given by extending normal modal logic K with the axiom $\mathsf{4} : \Diamond\Diamond a \to \Diamond a$. This correspondence can be generalised to *quasi-transitivity* (namely, if $x_i R x_{i+1}$ for $i \leq n-1$, then $x_0 R x_n$) by extending K with the axiom $\mathsf{4^n} : \Diamond^n a \to \Diamond a$ for $n \geq 3$.

This axiom is a specific case of modal reduction axioms $\Diamond^n a \to \Diamond^k a$ [1] as well as path axioms $\Diamond^n a \to \Box^l \Diamond a$ [4] or more generally of the Geach/Scott-Lemmon axioms $\Diamond^n \Box^m a \to \Box^l \Diamond^k a$ [5]. Several proof-theoretical treatments of these families of axioms have been considered, via labelled [8], nested [4] or indexed nested [7] sequents, to only cite a few examples, as well as comparison of these different approaches [3].

Here, we focus on the restricted family of quasi-transitive modal logics in the setting of *nested sequents* [2, 9]. Starting from nested sequent system nK, composed of rules id, $\wedge$, $\vee$, $\Box$ and $\Diamond_{\mathsf{k}}$, the transitive modal logic $\mathsf{K4}$ is sound complete with respect to $\mathsf{nK} + \Diamond_4$ (see Fig. 1). This can be proved, following [2], via a cut-elimination argument:

$$\begin{aligned} A \text{ is a theorem of } \mathsf{K}+\mathsf{4} &\iff A \text{ is provable in } \mathsf{nK} + \Diamond_4 + \mathsf{cut} \\ &\iff A \text{ is provable in } \mathsf{nK} + \Diamond_4 \end{aligned} \tag{1}$$

The cut-elimination result itself is a bit involved as it requires the introduction of a complex generalisation of the cut rule, called a $\mathsf{4cut}$.

On the other hand, [4] proves a soundness and completeness results in the general case of sets of path axioms, but by an external proof. If we specialise their result to any set of (quasi-)transitivity axioms $\mathsf{X} \subseteq \{\mathsf{4^n} \mid n \geq 2\}$, it can be stated as:

$$A \text{ is a theorem of } \mathsf{K}+\mathsf{X} \iff A \text{ is provable in } \mathsf{nK} + \{\Diamond_{\mathsf{kn}} \mid n \in \mathsf{comp_X}\} \text{ (Fig. 1)} \tag{2}$$

This specialisation to quasi-transitivity also allows us to simplify their notion of *completion*, namely, if we write $\mathsf{set_X}$ for the set $\{n \in \mathbb{N} \mid \mathsf{4^n} \in \mathsf{X}\}$, the set $\mathsf{comp_X}$ can be defined inductively as follows: $\mathsf{comp}_0 := \mathsf{set_X}$; $\mathsf{comp}_{p+1} := \mathsf{comp}_p \cup \{i+j-1 \mid i,j \in \mathsf{comp}_p\}$; and finally $\mathsf{comp_X} := \bigcup_{p=0}^{\infty} \mathsf{comp}_p$

We generalise (1) and provide a new proof of (2) which goes via an internal cut-elimination and interestingly lets us pinpoint precisely where the need for completion arises (see 2 ⇒ 3 in Thm. 1). We also provide an alternative nested sequent system to [4] which is in particular *modular*, that is, made of rules directly corresponding to each quasi-transitive axiom which can freely mix without requiring further rules unlike with completion (see 3 ⇒ 4 in Thm. 1).

THEOREM 1. *The following are equivalent:*

1. *A is a theorem of* $\mathsf{K}+\mathsf{X}$;
2. *A is provable in* $\mathsf{nK} + \Diamond_{\mathsf{kX}} + \mathsf{cut}$ *where* $\Diamond_{\mathsf{kX}} := \{\Diamond_{\mathsf{kn}} \mid n \in \mathsf{set_X}\}$;
3. *A is provable in* $\mathsf{nK} + \Diamond_{\mathsf{k\hat{X}}}$ *where* $\Diamond_{\mathsf{k\hat{X}}} := \{\Diamond_{\mathsf{kn}} \mid n \in \mathsf{comp_X}\}$;
4. *A is provable in* $\mathsf{nK} + \Diamond_{\mathsf{4X}}$ *where* $\Diamond_{\mathsf{4X}} := \{\Diamond_{\mathsf{4(n-1)}} \mid n \in \mathsf{set_X}\}$.

We give a sketch of the proof, full details can be found in [6]. For the direction 1 ⇒ 2, knowing that the axioms and rules of K are derivable using $\mathsf{nK} + \mathsf{cut}$, we only need to

$$
\mathsf{id}\ \frac{}{\Gamma\{a,\bar{a}\}}
\qquad
\wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}
\qquad
\vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}
\qquad
\Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}
\qquad
\mathsf{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\bar{A}\}}{\Gamma\{\varnothing\}}
$$

Modal propagation rules

$$
\Diamond_{\mathsf{k}}\ \frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}
\qquad
\Diamond_{\mathsf{kn}}\ \frac{\Gamma\{\Diamond A, [\Delta_1, [\ldots [\Delta_n, A] \ldots]]\}}{\Gamma\{\Diamond A, [\Delta_1, [\ldots [\Delta_n] \ldots]]\}}\ {\scriptstyle n \geq 1}
$$

$$
\Diamond_{\mathsf{4}}\ \frac{\Gamma\{\Diamond A, [\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}
\qquad
\Diamond_{\mathsf{4n}}\ \frac{\Gamma\{\Diamond A, [\Delta_1, [\ldots [\Delta_n, \Diamond A] \ldots]]\}}{\Gamma\{\Diamond A, [\Delta_1, [\ldots [\Delta_n] \ldots]]\}}\ {\scriptstyle n \geq 1}
$$

Modal structural rules

$$
\boxtimes_{\mathsf{k}}\ \frac{\Gamma\{[\Sigma], [\Delta]\}}{\Gamma\{[\Delta, \Sigma]\}}
\qquad
\boxtimes_{\mathsf{kn}}\ \frac{\Gamma\{[\Sigma], [\Delta_1, [\ldots, [\Delta_n] \ldots]]\}}{\Gamma\{[\Delta_1, [\ldots, [\Delta_n, \Sigma] \ldots]]\}}\ {\scriptstyle n \geq 1}
$$

FIGURE 1. Nested sequent rules

show that for any $n \geq 2$, axiom $\mathsf{4^n}$ is derivable using the rule $\Diamond_{\mathsf{kn}}$.

$$
\dfrac{\vee\ \dfrac{\Box\ \dfrac{\Diamond_{\mathsf{kn}}\ \dfrac{\mathsf{id}\ \dfrac{}{[\ldots [\bar{a}, a] \ldots], \Diamond a}}{[\ldots [\bar{a}] \ldots], \Diamond a}}{\Box^n \bar{a}, \Diamond a}\ {\scriptstyle n \text{ times}}}{\Box^n \bar{a} \vee \Diamond a}}{\mathsf{4^n} : \Diamond^n a \rightarrow \Diamond a}
$$

The direction $2 \Rightarrow 3$ is a cut-elimination theorem. By using the rules $\Diamond_{\mathsf{kn}}$ designed by [4], rather than the generalisation $\Diamond_{\mathsf{4n}}$ of the rule from [2], the cut-reduction case for the (quasi-)transitivity rules becomes simpler, without the need for a $\mathsf{4cut}$-style rule.

$$
\mathsf{cut}\ \dfrac{\Box\ \dfrac{\Gamma\{[A], [\Delta_1, [\ldots, [\Delta_n] \ldots]]\}}{\Gamma\{\Box A, [\Delta_1, [\ldots, [\Delta_n] \ldots]]\}} \qquad \Diamond_{\mathsf{kn}}\ \dfrac{\Gamma\{\Diamond \bar{A}, [\Delta_1, [\ldots [\Delta_n, \bar{A}] \ldots]]\}}{\Gamma\{\Diamond \bar{A}, [\Delta_1, [\ldots [\Delta_n] \ldots]]\}}}{\Gamma\{[\Delta_1, [\ldots [\Delta_n] \ldots]]\}}
$$

$$
\rightsquigarrow \quad \mathsf{cut}\ \dfrac{\boxtimes_{\mathsf{kn}}\ \dfrac{\Gamma\{[A], [\Delta_1, [\ldots, [\Delta_n] \ldots]]\}}{\Gamma\{[\Delta_1, [\ldots, [\Delta_n, A] \ldots]]\}} \qquad \Gamma\{[\Delta_1, [\ldots [\Delta_n, \bar{A}] \ldots]]\}}{\Gamma\{[\Delta_1, [\ldots [\Delta_n] \ldots]]\}}
$$

For the left premiss, we need to show that for any $n \in \mathsf{set_X}$ the structural rule $\boxtimes_{\mathsf{kn}}$ is admissible in $\mathsf{nK} + \Diamond_{\mathsf{k\hat{X}}}$. The right premiss is obtained by induction on proof height.

For direction $3 \Rightarrow 4$, we need to show that for $n \in \mathsf{comp_X}$, the rules $\Diamond_{\mathsf{kn}}$ and $\Diamond_{\mathsf{4(n-1)}}$ are derivable in $\mathsf{nK} + \Diamond_{\mathsf{4X}}$ by induction on the definition of $\mathsf{comp_X}$. As a matter of example, if $n \in \mathsf{comp}_{p+1}$ and $n = i + j - 1$ for some $i, j \in \mathsf{comp}_p$, by induction hypothesis, $\Diamond_{\mathsf{4(i-1)}}$ and $\Diamond_{\mathsf{kj}}$ are derivable, hence $\Diamond_{\mathsf{kn}}$ can be shown derivable as follows:

$$
\Diamond_{\mathsf{4(i-1)}}\ \dfrac{\Diamond_{\mathsf{kj}}\ \dfrac{\Gamma\{\Diamond A, [\Delta_1, [\ldots [\Delta_{i+j-1}, A] \ldots]]\}}{\Gamma\{\Diamond A, [\Delta_1, [\ldots [\Delta_{i-1}, \Diamond A, [\ldots [\Delta_{i+j-1}] \ldots]] \ldots]]\}}}{\Gamma\{\Diamond A, [\Delta_1, [\ldots [\Delta_{i+j-1}] \ldots]]\}}
$$

Finally, the direction $4 \Rightarrow 1$ is simply stating the soundness of rules in $\mathsf{nK} + \Diamond_{\mathsf{4X}}$.

We conjecture the approach used here of *propagating* formulas $\Diamond A$ to ease the requirement of completion given in [4] could be generalised to general path axioms.

[1] Johan van Benthem, *Modal reduction principles*, ***The Journal of Symbolic Logic***, vol. 41 (1976), no. 2, pp. 301–312.

[2] Kai Brünnler, *Deep sequent systems for modal logic*, ***Archive for Mathematical Logic***, vol. 48 (2009), no. 6, pp. 551–577.

[3] Agata Ciabattoni, Tim Lyon, Revantha Ramanayake, and Alwen Tiu, *Display to labeled proofs and back again for tense logics*, ***ACM Transactions on Computational Logic***, vol. 22 (2021), no. 3, pp. 1–31.

[4] Rajeev Goré, Linda Postniece, and Alwen Tiu, *On the correspondence between display postulates and deep inference in nested sequent calculi for tense logics*, ***Logical Methods in Computer Science***, vol. 7 (2011), no. 2:8, pp. 1–38.

[5] Edward J. Lemmon and Dana S. Scott, ***An introduction to modal logic***, Blackwell, 1977.

[6] Sonia Marin and Paaras Padhiar, *Nested sequents for quasi-transitive modal logics*, arXiv:2406.09300, 2024

[7] Sonia Marin and Lutz Strassburger, *Proof theory for indexed nested sequents*, ***Automated Reasoning with Analytic Tableaux and Related Methods*** (Cham), (Renate A Schmidt and Cláudia Nalon, editors), Springer International Publishing, 2017, pp. 81–97.

[8] Sara Negri, *Proof analysis in modal logics*, ***Journal of Philosophical Logic***, vol. 34 (2005), no. 5–6, pp. 507–544.

[9] Francesca Poggiolesi, ***Gentzen calculi for modal propositional logic***, vol. 32, Springer Science & Business Media, 2010.