Nested Proof Theory for Quasi-Transitive Modal Logics

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Abstract

Previous works by Goré, Postniece and Tiu have provided sound and cut-free complete proof systems for modal logics extended with path axioms using the formalism of nested sequent. Our aim is to provide (i) an internal cut-elimination procedure and (ii) alternative modular formulations for these systems. We present our methodology to achieve these two goals on a subclass of path axioms, namely quasi-transitivity axioms, and discuss how it could be extended further to quasi-symmetry axioms.

Keywords: Path axioms, Proof theory, Nested sequents, Cut-elimination, Modularity.

1 Introduction

The proof theory of modal logics has been explored thoroughly and many authors have contributed to the deep understanding gathered to this day. In particular, it has been remarked time and time again that in order to capture the validities of a modal logic, additional structure, often inspired by the semantics of the logic itself, is required within the proof-theoretical syntax. This led to the development of many formalisms extending Gentzen's sequent calculus, such as hypersequents [1], nested sequents [2,9], and labelled sequents [8].

It is not always clear however what sort of additional structure is precisely required to design the proof theory of a modal logic. For example, modal logic $\mathsf{S5}$ can be expressed using labelled or nested sequents, but can also be given a sound and complete system in the lighter hypersequent formalism, whereas such a result is conjectured not to be possible in ordinary sequent calculus [5].

Goré, Postniece and Tiu [4] have proposed a general algorithm to design nested sequent systems for modal logic K [1] extended with *path axioms*, of the

[1] Their work takes place in the context of tense logic where the language contains also adjoint

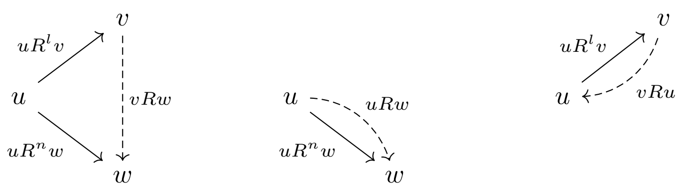

Fig. 1. Frame conditions for path, quasi-transitivity, and quasi-symmetry axioms

form $\Diamond^n a \supset \Box^l \Diamond a$. As a subclass of Scott-Lemmon axioms $\Diamond^n \Box^m a \supset \Box^l \Diamond^k a$ [6], they enjoy a well-behaved correspondence with the frame conditions displayed on the left of Fig. 1, i.e., if $uR^l v$ and $uR^n w$ then vRw. They are also known to be decidable logics, as shown by [4] using automata theoretic methods.

In this line of work, we set out to understand more precisely Goré et al.'s systems proof theoretically, in particular on a methodolgy to (i) equip them with an internal *cut-elimination* procedure and (ii) distill them into *modular* systems, i.e., such that each axiom corresponds to a (set of) rule(s) which can be freely mixed with others independently of the other axioms present.

We started with some restricted classes of path axioms. We call them *quasi-transitivity* when $l = 0$, giving $\mathsf{4^n} : \Diamond^n a \supset \Diamond a$ for $n \geq 1$, and *quasi-symmetry* when $n = 0$, giving $\mathsf{b^l} : a \supset \Box^l \Diamond a$ for $l \geq 1$. These correspond respectively to the frame conditions displayed in the middle and on the right of Fig. 1.

In this short paper we present our preliminary results regarding quasi-transitive modal logics and discuss the difficulty we faced so far to export the approach to quasi-symmetric ones.

2 Nested sequent for quasi-transitive logics

2.1 Nested sequents

A *nested sequent* is defined as $\Gamma ::= \varnothing \mid A, \Gamma \mid \Gamma, [\Gamma]$ which corresponds informally to a tree of sequents and can be expressed inductively in the modal language as: $\mathsf{form}(\varnothing) = \bot$, $\mathsf{form}(A, \Gamma) = A \vee \mathsf{form}(\Gamma)$, and $\mathsf{form}(\Gamma_1, [\Gamma_2]) = \mathsf{form}(\Gamma_1) \vee \Box \mathsf{form}(\Gamma_2)$.

A *context* is a nested sequent with one or several holes $\{\ \}$ which can take the place of a formula in the sequent (but does not occur inside a formula). This lets us write $\Gamma_1\{\Gamma_2\}$ when we replace the hole in $\Gamma_1\{\ \}$ by Γ_2.

Nested sequent system nK is composed of the rules id, $\wedge$, $\vee$, $\Box$ and $\Diamond_\mathsf{k}$ in Fig. 2 and is sound and complete wrt. K [2]. We can further extend this system with some of the modal propagation or modal structural rules in Fig. 2 to capture wider modal logics. The modal rules on the LHS ($\Diamond_\mathsf{k}$, $\Diamond_\mathsf{4}$, $\Diamond_\mathsf{b}$, as well as their structural versions) are taken from [2]. The ones on the RHS are the generalisations we are studying here, including $\Diamond_\mathsf{kn}$ which appears in [4].

A *proof* is then built from these rules as a tree in the same way as in

modalities, but we restrict our attention to the language with only $\Box$ and $\Diamond$.

$$\mathsf{id}\ \frac{}{\Gamma\{a,\bar{a}\}} \qquad \wedge\ \frac{\Gamma\{A\}\quad \Gamma\{B\}}{\Gamma\{A\wedge B\}} \qquad \vee\ \frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}} \qquad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \mathsf{cut}\ \frac{\Gamma\{A\}\quad \Gamma\{\bar{A}\}}{\Gamma\{\varnothing\}}$$

Modal propagation rules

$$\Diamond_{\mathsf{k}}\ \frac{\Gamma\{\Diamond A,[\Delta,A]\}}{\Gamma\{\Diamond A,[\Delta]\}} \qquad \Diamond_{\mathsf{kn}}\ \frac{\Gamma\{\Diamond A,[\Delta_1,[\ldots,[\Delta_n,A]\ldots]]\}}{\Gamma\{\Diamond A,[\Delta_1,[\ldots,[\Delta_n]\ldots]]\}}\ n\geq 1$$

$$\Diamond_{\mathsf{4}}\ \frac{\Gamma\{\Diamond A,[\Delta,\Diamond A]\}}{\Gamma\{\Diamond A,[\Delta]\}} \qquad \Diamond_{\mathsf{4n}}\ \frac{\Gamma\{\Diamond A,[\Delta_1,[\ldots,[\Delta_{n-1},\Diamond A]\ldots]]\}}{\Gamma\{\Diamond A,[\Delta_1,[\ldots,[\Delta_{n-1}]\ldots]]\}}\ n\geq 1$$

$$\Diamond_{\mathsf{b}}\ \frac{\Gamma\{A,[\Delta,\Diamond A]\}}{\Gamma\{[\Delta,\Diamond A]\}} \qquad \Diamond_{\mathsf{bn}}\ \frac{\Gamma\{A,[\Delta_1,[\ldots,[\Delta_n,\Diamond A]\ldots]]\}}{\Gamma\{[\Delta_1,[\ldots,[\Delta_n,\Diamond A]\ldots]]\}}\ n\geq 1$$

Modal structural rules

$$\boxtimes_{\mathsf{k}}\ \frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[\Delta,\Sigma]\}} \qquad \boxtimes_{\mathsf{kn}}\ \frac{\Gamma\{[\Sigma],[\Delta_1,[\ldots,[\Delta_n]\ldots]]\}}{\Gamma\{[\Delta_1,[\ldots,[\Delta_n,\Sigma]\ldots]]\}}\ n\geq 1$$

$$\boxtimes_{\mathsf{b}}\ \frac{\Gamma\{[\Delta,[\Sigma]]\}}{\Gamma\{\Sigma,[\Delta]\}} \qquad \boxtimes_{\mathsf{bn}}\ \frac{\Gamma\{[\Delta_1,[\ldots,[\Delta_i,[\Sigma]]\ldots]]\}}{\Gamma\{\Sigma,[\Delta_1,[\ldots,[\Delta_i]\ldots]]\}}\ n\geq 1$$

Fig. 2. Nested sequent rules

ordinary sequent calculi. A rule $\frac{\Gamma_1}{\Gamma_2}$ is *admissible* in a nested sequent system N if whenever there is a proof of Γ_1 in N, there is a proof of Γ_2 in N.

2.2 Quasi-transitive modal logics

Let us look first at the transitive modal logic $\mathsf{K4}$, which is known to be sound complete with respect to $\mathsf{nK}+\Diamond_{\mathsf{4}}$ (see Fig. 2). This can be proved, following Brünnler [2], via a cut-elimination argument, that is:

$$\begin{aligned} A \text{ is a theorem of } \mathsf{K}+\mathsf{4} &\iff A \text{ is provable in } \mathsf{nK}+\Diamond_{\mathsf{4}}+\mathsf{cut} \\ &\iff A \text{ is provable in } \mathsf{nK}+\Diamond_{\mathsf{4}} \end{aligned} \tag{1}$$

The cut-elimination proof itself is a bit involved as it requires the introduction of a complex generalisation of the cut rule, called a $\mathsf{4cut}$ in [2]

Meanwhile, Goré, Postniece and Tiu [4] prove a general soundness and completeness result for sets of path axioms, which we specialise to sets of quasi-transitivity axioms.

Let X be from now on be a subset of the positive natural numbers. We will write $\mathsf{K}+\mathsf{4}^{\mathsf{X}}$ to denote the modal logic K extended with (quasi-)transitivity axioms $\mathsf{4}^{\mathsf{n}}:\Diamond^n a\supset\Diamond a$ for each $n\in X$ and use the notations $\Diamond_{\mathsf{kX}}:=\{\Diamond_{\mathsf{kn}}:n\in X\}$ and $\Diamond_{\mathsf{4X}}:=\{\Diamond_{\mathsf{4n}}:n\in X\}$ (rules in Fig. 2).

The specialisation to quasi-transitivity allows us to simplify the notion of *completion*, namely, the set $\hat{X}$ can be defined inductively as:

$$X_0:=X \qquad X_{p+1}:=X_p\cup\{i+j-1\mid i,j\in X_p\} \qquad \hat{X}:=\bigcup_{p=0}^{\infty}X_p \tag{2}$$

This is a simplification of the completion given for a set of path axioms in [4], which is calculated from the algebra of paths for the propagation graph of a nested sequent. It allows Goré et al. to establish certain sets of $\Diamond_{\mathsf{kn}}$ rules to be sound and complete for certain path axioms.

Once simplified to quasi-transitive logics, their result can then be stated as:

$$A \text{ is a theorem of } \mathsf{K} + 4^{\mathsf{X}} \iff A \text{ is provable in } \mathsf{nK} + \Diamond_{\mathsf{k}\hat{\mathsf{X}}} \tag{3}$$

However, the obtained systems are not *modular* as for two subsets of positive natural numbers X_1, X_2, the completion of $X_1 \cup X_2$ is not generally $\hat{X}_1 \cup \hat{X}_2$.

2.3 Main result and proof sketch

We generalise (1) and give an alternative proof of (3), via cut-elimination, to refine soundness and completeness for quasi-transitive modal logics:

Theorem 2.1 *The following are equivalent:*

(i) *A is a theorem of* $\mathsf{K} + 4^{\mathsf{X}}$

(ii) *A is provable in* $\mathsf{nK} + \Diamond_{\mathsf{kX}} + \mathsf{cut}$

(iii) *A is provable in* $\mathsf{nK} + \Diamond_{\mathsf{k}\hat{\mathsf{X}}}$

(iv) *A is provable in* $\mathsf{nK} + \Diamond_{\mathsf{4X}}$

We give a sketch of the proof here; the full version can be found in [7].

For the direction (i) $\Rightarrow$ (ii), knowing that the axioms and rules of K are derivable using $\mathsf{nK} + \mathsf{cut}$, we only need to show that for any $n \geq 1$, axiom 4^{n} is derivable using the corresponding rule $\Diamond_{\mathsf{kn}}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[\ldots[\bar{a}, a]\ldots], \Diamond a}\ \mathsf{id}}{[\ldots[\bar{a}]\ldots], \Diamond a}\ \Diamond_{\mathsf{kn}}}{\Box^n \bar{a}, \Diamond a}\ \Box\ n \text{ times}}{\Box^n \bar{a} \vee \Diamond a}\ \vee}{\Diamond^n a \supset \Diamond a}$$

The direction (ii) $\Rightarrow$ (iii) is a cut-elimination proof. The result can be deduced via the proof translations from [4]; we provide here an internal proof which moreover lets us pinpoint precisely where the need for completion arises. By using the rules $\Diamond_{\mathsf{kn}}$ designed by [4], rather than the generalisation $\Diamond_{\mathsf{4n}}$ of the rule from [2], the cut-reduction case for the quasi-transitivity rules becomes much simpler, without the need to consider a $\mathsf{4cut}$-style rule. The cut-elimination argument follows as usual and most of the cases are identical to [2]. We are left to consider a cut of the form:

$$\dfrac{\dfrac{\Gamma\{[A], [\Delta_1, [\ldots, [\Delta_n]\ldots]]\}}{\Gamma\{\Box A, [\Delta_1, [\ldots, [\Delta_n]\ldots]]\}}\ \Box \qquad \dfrac{\Gamma\{\Diamond\bar{A}, [\Delta_1, [\ldots, [\Delta_n, \bar{A}]\ldots]]\}}{\Gamma\{\Diamond\bar{A}, [\Delta_1, [\ldots, [\Delta_n]\ldots]]\}}\ \Diamond_{\mathsf{kn}}}{\Gamma\{[\Delta_1, [\ldots, [\Delta_n]\ldots]]\}}\ \mathsf{cut}$$

which can be reduced to a cut on smaller formulas A and $\bar{A}$

$$\mathsf{cut}\ \dfrac{\boxtimes_{\mathsf{kn}}\ \dfrac{\Gamma\{[A],[\Delta_1,[\ldots,[\Delta_n]\ldots]]\}}{\Gamma\{[\Delta_1,[\ldots,[\Delta_n,A]\ldots]]\}} \quad \Gamma\{[\Delta_1,[\ldots,[\Delta_n,\bar{A}]\ldots]]\}}{\Gamma\{[\Delta_1,[\ldots,[\Delta_n]\ldots]]\}}$$

where the right premiss is obtained by applying the induction hypothesis on height to a cut on $\Box A$ and $\Diamond \bar{A}$ and the left premiss, is obtained by admissibility of the modal structural rules, namely:

Lemma 2.2 *For each $n \in X$, $\boxtimes_{\mathsf{kn}}$ is admissible in* $\mathsf{nK} + \Diamond_{\mathsf{k}\hat{\mathsf{X}}}$.

In the proof of this lemma, the requirement for completion becomes apparent.

The direction (iii) $\Rightarrow$ (iv) is where we achieve modularity. Indeed, using the $\Diamond_{\mathsf{4n}}$ rule which *propagates* formulas $\Diamond A$ (similar to the $\Diamond_{\mathsf{4}}$ rule from [2]), rather than the $\Diamond_{\mathsf{kn}}$ rule, allows us to drop the requirement of completion.

We need to show that for $n \in \hat{X}$, the rules $\Diamond_{\mathsf{kn}}$ and $\Diamond_{\mathsf{4n}}$ are derivable in $\mathsf{nK} + \Diamond_{\mathsf{4X}}$ by induction on the definition of $\hat{X}$. As a matter of example, if $n \in X_{p+1}$ and $n = l + m - 1$ for some $l, m \in X_p$, by induction hypothesis $\Diamond_{\mathsf{4l}}$ and $\Diamond_{\mathsf{km}}$ are derivable, hence $\Diamond_{\mathsf{kn}}$ can be shown derivable:

$$\Diamond_{\mathsf{k(l+m)}}\ \dfrac{\Gamma\{\Diamond A,[\Delta_1,[\ldots,[\Delta_{l+m},A]\ldots]]\}}{\Gamma\{\Diamond A,[\Delta_1,[\ldots,[\Delta_{l+m}]\ldots]]\}}$$

$$\equiv\ \Diamond_{\mathsf{4l}}\ \dfrac{\Diamond_{\mathsf{km}}\ \dfrac{\Gamma\{\Diamond A,[\Delta_1,[\ldots,[\Delta_{l+m},A]\ldots]]\}}{\Gamma\{\Diamond A,[\Delta_1,[\ldots,[\Delta_{l-1},\Diamond A,[\ldots,[\Delta_{l+m}]]\ldots]\ldots]]\}}}{\Gamma\{\Diamond A,[\Delta_1,[\ldots,[\Delta_{l+m}]\ldots]]\}}$$

Finally, the direction (iv) $\Rightarrow$ (i) is stating the soundness of rules in $\mathsf{nK}+\Diamond_{\mathsf{4X}}$. The soundness of the rules in the system nK are proved in [2]. For a rule $\Diamond_{\mathsf{4n}}\frac{\Gamma_1}{\Gamma_2}$, we can similarly show that $\mathsf{form}(\Gamma_1) \supset \mathsf{form}(\Gamma_2)$ is a theorem of $\mathsf{K} + \mathsf{4}^{\mathsf{X}}$.

Note that the nested sequent systems we considered in this section, for quasi-transitive logics, are exclusively *propagation rule* based. The *structural rules* are used in the process of the cut-elimination proof but do not need to be explicitly added to the systems.

3 Towards nested sequents for quasi-symmetric logics

The completion we gave in (2) is a simplification of the completion given for a set of path axioms in [4], which is calculated by looking at the algebra of paths for a propagation graph of a nested sequent.

When specialising it again for the quasi-symmetric axioms $\mathsf{b}^{\mathsf{l}} : a \supset \Box^l \Diamond a$, we get the rules $\Diamond_{\mathsf{bn}}$ in Fig. 2. However, we have so far been unable to replicate the methodology developed in the previous section for this set of rules. In particular, the approach is unsuccessful when attempting to prove $\boxtimes_{\mathsf{bl}}$ admissible.

The strategy, similar to Lemma 2.2, would be to prove this through an induction on the height of the proof. That is, for example when $i = 2$, we would like to transform the derivation

$$\boxtimes_{\mathsf{b2}}\dfrac{\Diamond_{\mathsf{b2}}\dfrac{\Gamma\{[\Delta_1,[\Delta_2,A,[\Sigma_1,[\Sigma_2,\Diamond A]]]]\}}{\Gamma\{[\Delta_1,[\Delta_2,[\Sigma_1,[\Sigma_2,\Diamond A]]]]\}}}{\Gamma\{\Sigma_1,[\Sigma_2,\Diamond A],[\Delta_1,[\Delta_2]]\}}$$

into a derivation of this shape

$$\dfrac{\boxtimes_{\mathsf{b2}}\dfrac{\Gamma\{[\Delta_1,[\Delta_2,A,[\Sigma_1,[\Sigma_2,\Diamond A]]]]\}}{\Gamma\{\Sigma_1,[\Sigma_2,\Diamond A],[\Delta_1,[\Delta_2,A]]\}}}{\Gamma\{\Sigma_1,[\Sigma_2,\Diamond A],[\Delta_1,[\Delta_2]]\}}$$

Unfortunately, the step indicated as a dashed line does not seem to correspond to any of the rules considered in this paper. It is not clear at this point whether the completion in [4] is enough to perform this step.

An alternative approach, inspired by Brünnler and Straßburger's [3], would be to renounce modal propagation rules and try to obtain a cut-elimination result for a system based exclusively on modal structural rules. These different avenues are the subject of ongoing work.

4 Concluding remarks

In this work, we proposed a proof-theoretic study of nested sequent systems for path axioms [4]. We showed its benefits on the special case of quasi-transitivity axioms and how it is challenged by quasi-symmetry axioms. The interest of the approach is not strictly to provide new nested sequent systems to a restricted family of logics for which we already know sound and complete systems in many different formalisms, including nested sequents. It is rather to dive deeper into some of these existing systems to understand the reason behind non-modularity and to ease the requirement of completion for path axioms in general.

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