Justification Logic for Intuitionistic Modal Logic

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Abstract. Justification logic is an explication of modal logic: boxes are replaced with proof terms formally through realisation theorems. This can be achieved syntactically using a cut-free proof system for a modal logic, e.g., using sequent, hypersequent, or nested sequent calculi. In constructive modal logic, boxes and diamonds are decoupled and not De Morgan dual. Previous work provides a justification counterpart to constructive modal logic CK (and some extensions) by making diamonds explicit and introducing new terms called satisfiers. We continue this line of work and provide a justification counterpart to intuitionistic modal logic IK and its extensions with the t and 4 axioms. We extend the syntax of proof terms to accommodate the additional axioms of intuitionistic modal logic and provide an axiomatisation of these justification logics with a syntactic realisation procedure using a cut-free nested sequent system for intuitionistic modal logic.

Keywords: Justification logic · Intuitionistic modal logic · Realisation theorem · Nested sequents

1 Introduction

Justification logics are a family of logics which refine modal logics by replacing modal operators with explicit *proof terms*. In a similar way to how a modal formula $\Box A$ can be read as *A is provable* in provability logics, it can be given an explicit justification counterpart $t{:}A$ for some proof term t, to be read as *there exists a proof t of A*.

The first such justification logic is the Logic of Proofs, LP, introduced by Artemov [2,3] which is an explicitation of the modal logic S4. Artemov showed that this logic, which can be seen as provability semantics to intuitionistic logic (IPL) via the Gödel translation of IPL into S4, enjoys an arithmetic interpretation into Peano Arithmetic (PA). The formal connection between LP and S4 is made through a *realisation theorem*, which translates each theorem of S4 into a corresponding theorem of LP by *realising* every modalities with proof terms. The first realisation theorem was proved syntactically by Artemov using a cut-free sequent calculus for S4 and Fitting later provided an alternative semantic method [22].

LP was also further generalised, and realisation theorems were obtained for instance for the modal S5 cube and the family of Geach logics (see [7] for a

survey) as Artemov's method for realisation can be adapted to modal logics if they have a cut-free sequent calculus (see [29]). Moreover, this syntactic method was expanded to more exotic cut-free proof systems: based on the formalism of hypersequents [9] and of nested sequents [15,25,12].

Since the early days of justification logic, there has been an interest in their intuitionistic variants, originally as a way to unify the treatment of terms between LP and the λ-calculus [4]. Another line of research that has brought justification logic into the realm of intuitionistic modal logics, is the investigation of the provability logic of Heyting Arithmetic (HA). Some propositions for an intuitionistic version of LP have been put forward in [8,18], as a building block towards a justification logic corresponding to HA in the way that LP corresponds to PA. Other intuitionistic variants of justification logics have been considered as a basis for type systems for computations that have access to some of their execution [44]. The semantics and the proof theory of some of these logics have also been partially studied [36,37,27]; however none of these work provide counterparts to an intuitionistic modal logic which contains both the box and the diamond modalities.

The first justification logic for an intuitionistic modal logic which makes the diamond explicit was postulated in [30] as a justification counterpart to *constructive modal logic* CK [42,11,10] via a syntactic realisation procedure. As $\Box$ is replaced by a proof term t; so is the diamond operator $\Diamond$ replaced with a *satisfier* term μ.

We expand this line of work to provide a justification counterpart to the intuitionistic variant of modal logic IK, as defined originally by [19,20,21,39]. Interestingly, intuitionistic modal logic IK is remarkably more expressive than constructive modal logic CK as it not only provides more validities about the $\Diamond$ operator (in particular, that it is normal), but it also permits the derivation of more $\Diamond$-free theorems [17]. One such theorem is the Gödel-Gentzen (double-negation) translation, which makes for example the relationship of IK with classical K more akin to the relationship of HA with PA than could be obtained with CK, and is also more amenable to functional interpretations.

We present an axiomatisation of our justification counterpart to IK and extensions with the t and 4 axioms and establish a formal connection through a syntactic realisation procedure using a cut-free system for IK and extensions [45] using nested sequents [16,28,13,14,40,41].

2 Modal and Justification Logics

2.1 Intuitionistic Modal Logic

Formulas of intuitionistic modal logic, $\mathcal{L}_{\Box}$, are given by the grammar:

$$A ::= \bot \mid p \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid \Box A \mid \Diamond A$$

where p ranges over a countable set of propositional atoms Prop. We will drop the outermost brackets when it is not required. We will assume $\wedge$ and $\vee$ are associative, $\wedge$ is commutative, and $\to$ is right associative.

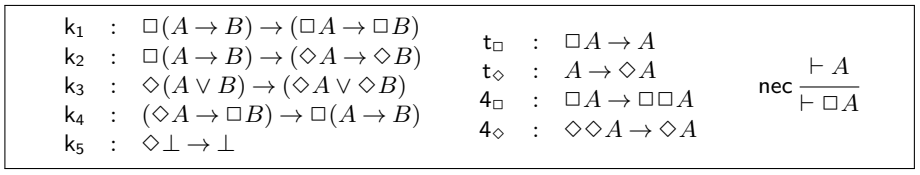

Fig. 1. Intuitionistic modal axioms

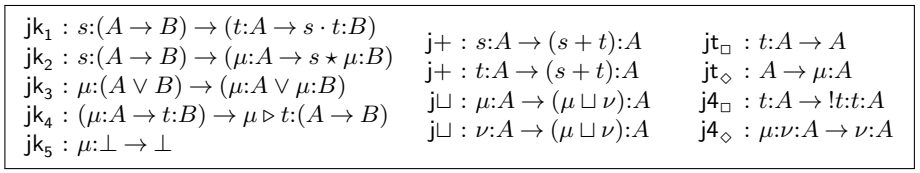

Fig. 2. Intuitionistic justification axioms

On Fig. 1 we list the modal axioms that we consider in this work. *Constructive* modal logic CK [10] is an extension of intuitionistic propositional logic IPL with axioms k_1 and k_2 and the necessitation inference. *Intuitionistic* modal logic IK [19,20,21,39] is an extension of CK with axioms k_3, k_4 and k_5. We will also study extensions of IK: with axioms $\mathsf{t}_\Box$ and $\mathsf{t}_\Diamond$ to logic IT, with axioms $\mathsf{4}_\Box$ and $\mathsf{4}_\Diamond$ to logic $\mathsf{IK4}$, and finally to logic $\mathsf{IS4}$ with all four axioms.

2.2 Justification Logic

Formulas of intuitionistic justification logic, $\mathcal{L}_\mathsf{J}$, are given by the grammar:

$$A ::= \bot \mid p \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid t{:}A \mid \mu{:}A$$

where p ranges over a countable set propositional atoms Prop, t ranges over *proof terms* PrfTm and μ ranges over *satisfiers* SatTm.

Proof terms $t, s, \ldots$ and *satisfiers* $\mu, \nu, \ldots$ are generated as follows:

$$t ::= x, y \mid c \mid (t+t) \mid (t \cdot t) \mid (\mu \rhd t) \mid !t$$
$$\mu ::= \alpha, \beta \mid (\mu \sqcup \mu) \mid (t \star \mu)$$

where x, y range over proof variables PrfVar, α, β over satisfier variables SatVar, and c over proof constants $\mathsf{PrfConst}$. A term is called *ground* if it contains neither proof nor satisfier variables. For a formula A, we write $\mathrm{var}(A)$ for the set $\{x \in \mathsf{PrfVar} \mid x \text{ occurs in A}\} \cup \{\alpha \in \mathsf{SatVar} \mid \alpha \text{ occurs in A}\}$.

We define the *justification counterpart of modal logic* IK, which we call JIK, as the extension of IPL with axioms $\mathsf{jk}_1 - \mathsf{jk}_5$, $\mathsf{j+}$ and $\mathsf{j\sqcup}$ on Fig. 2, together with the *constant axiom necessitation rule*:

$$\mathsf{can}\ \frac{}{c_n{:}\ldots{:}c_1{:}A}$$

where $c_1, \ldots, c_n \in \mathsf{PrfConst}$ and A is any axiom instance in Fig. 2.

We furthermore define justification counterparts to the extensions of IK introduced previously using the additional axioms on Fig. 2:

$$\mathsf{JIT} = \mathsf{JIK} + \mathsf{jt}_\Box + \mathsf{jt}_\Diamond \quad \mathsf{JIK4} = \mathsf{JIK} + \mathsf{j4}_\Box + \mathsf{j4}_\Diamond \quad \mathsf{JIS4} = \mathsf{JIK} + \mathsf{jt}_\Box + \mathsf{jt}_\Diamond + \mathsf{j4}_\Box + \mathsf{j4}_\Diamond$$

The operations *proof sum* $+$, *application* $\cdot$ and *proof checker* ! are the standard justification operations relating to proof manipulations. To build intuition for the application operator, recall the jk_1 axiom $s{:}(A \to B) \to (t{:}A \to s \cdot t{:}B)$. It internalises *modus ponens*, namely that a proof t of A and a proof s of $A \to B$ can be composed to obtain a proof $s \cdot t$ of B.

The proof terms embed an abstract notion of *global reasoning* in the way that they assert the *validity* of statements wrt. any model. Satisfiers give us a seemingly dual notion of *local reasoning*, where $\mu{:}A$ could be read as μ is a *model* where A is satisfied, or that μ asserts the *consistency* of A.

The operations on satisfiers of *propagation* $\star$ and *disjoint union* $\sqcup$ were introduced in [30]. The operation $\star$, which builds a satisfier from a proof term and a satisfier, can be seen as a combination of local and global reasoning, e.g., suppose that we have $\mu{:}A$, we can use the proof t of $A \to (A \vee B)$ to locally establish $t \star \mu{:}(A \vee B)$ using axiom jk_2. While $A \vee B$ is true whenever A is, the justification used is different in that the former involves a valid transition from A to $A \vee B$ justified by t. Hence, instead of using the same satisfier μ, we record our reasoning in the new satisfier $t \star \mu$.

The operation $\sqcup$ is akin to that of a disjoint union of models. For sets, disjoint union is often defined $X \sqcup Y := (X \times \{0\}) \cup (Y \times \{1\})$ to create separate copies of X and Y to avoid intersection. We adopt the same notion to our disjoint model union. Any overlaps of μ and ν are resolved before the models are combined and no connection between the μ and ν parts of the satisfier $\mu \sqcup \nu$ exists.

In this work, we introduce the new operation of *local update* $\triangleright$ which carries the intuition that if local reasoning implies global reasoning, one can update this global reasoning with the local. Restating the jk_4 axiom here, $(\mu{:}A \to t{:}B) \to \mu \triangleright t{:}(A \to B)$, the satisfier μ locally reasons about A and implies the proof term t globally reasoning about B, so this notion is *updated* into global reasoning about $A \to B$ being kept track by the proof term $\mu \triangleright t$.

2.3 From JL to L – and back

From now, we will write L for any logic in $\{\mathsf{IK}, \mathsf{IT}, \mathsf{IK4}, \mathsf{IS4}\}$ and JL for the corresponding logic in $\{\mathsf{JIK}, \mathsf{JIT}, \mathsf{JIK4}, \mathsf{JIS4}\}$. The main point of this work is to establish a syntactic correspondence between the logics introduced in previous sections.

Definition 1 (Forgetful projection). *The forgetful projection is a map*

$$(\cdot)^{\mathsf{f}} : \mathcal{L}_{\mathsf{J}} \to \mathcal{L}_{\Box}$$

inductively defined as follows:

$$\bot^{\mathsf{f}} := \bot \qquad (t{:}A)^{\mathsf{f}} := \Box A^{\mathsf{f}}$$
$$p^{\mathsf{f}} := p \qquad (\mu{:}A)^{\mathsf{f}} := \Diamond A^{\mathsf{f}}$$
$$(A * B)^{\mathsf{f}} := (A^{\mathsf{f}} * B^{\mathsf{f}}) \text{ where } * \in \{\wedge, \vee, \to\}$$

Theorem 2. *Let $A \in \mathcal{L}_{\mathsf{J}}$. If $\mathsf{JL} \vdash A$ then $\mathsf{L} \vdash A^{\mathsf{f}}$.*

Proof. This follows from the fact that the forgetful projection on axioms of JL and conclusions of the can rule are theorems of L.

Definition 3 (Realisation). *A realisation is a map $(\cdot)^r : \mathcal{L}_{\Box} \to \mathcal{L}_{\mathsf{J}}$ such that $(A^r)^{\mathsf{f}} = A$ for each $A \in \mathcal{L}_{\Box}$.*

Theorem 4. *Let $A \in \mathcal{L}_{\Box}$. If $\mathsf{L} \vdash A$, then there exists a realisation r such that $\mathsf{JL} \vdash A^r$.*

The goal of this paper is to prove this theorem. We first need to introduce the main technical ingredients of the proof.

2.4 Realisation over Annotations

A subformula of A is *positive* if its position in the formula tree of A is reached from the root by following the left branch of an $\to$ an even number of times; otherwise it is called *negative*.

A realisation is *normal* if when $t : B$ (resp. $\mu : B$) is a negative subformula of A^r, then $t \in \mathsf{PrfVar}$ (resp. $\mu \in \mathsf{SatVar}$) and occurs in A exactly once.

We follow and expand the methodology introduced in [25] using nested sequents to prove a realisation theorem. Due to the complexity of the sequent structure, some extra machinery is required to ensure a careful bookkeeping during the proof manipulations.

Definition 5 (Annotated formula). Annotated formulas *are built as in $\mathcal{L}_{\Box}$ but using annotated boxes $\Box_n$ and diamonds $\Diamond_n$ for $n \in \mathbb{N}$.*
For an annotated formula A, define $ann(A) := \{n \in \mathbb{N} \mid \Box_n$ or $\Diamond_n$ occurs in $A\}$.

Definition 6 (Properly annotated). *A formula A is* properly annotated *if:*

- *modalities are annotated by pairwise distinct indexes;*
- *a $\Box$ (resp. $\Diamond$) is indexed by n iff $n = 0, 1 \mod 4$ (resp. $n = 2, 3 \mod 4$);*
- *a modality indexed by n occurs in a positive (resp. negative) position iff n is even (resp. odd).*

The set of properly annotated formulas is denoted $\mathcal{L}_{\Box}^{\mathsf{ann}}$.

For $\mathsf{L} \in \{\mathsf{IK}, \mathsf{IT}, \mathsf{IK4}, \mathsf{IS4}\}$, we will write $\mathsf{L} \vdash A$ for $A \in \mathcal{L}_{\Box}^{\mathsf{ann}}$, when the unannotated version of A is a theorem of L.

We assume that we have fixed an enumeration of proof variables $\mathsf{PrfVar} = \{x_1, x_2, \ldots, y_1, y_2, \ldots\}$ and of satisfier variables $\mathsf{SatVar} = \{\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots\}$.

Definition 7 (Realisation on annotations). *A* realisation on annotations *is a partial function*

$$r : \mathbb{N} \to \mathsf{PrfTm} \cup \mathsf{SatTm}$$

where $r(4n) \in \mathsf{PrfTm}$, $r(4n+1) = x_n$, $r(4n+2) \in \mathsf{SatTm}$, $r(4n+3) = \alpha_n$ when defined. Denote the domain of r as $dom(r) := \{n \in \mathbb{N} \mid r(n)$ is defined$\}$.

Given $ann(A) \subseteq dom(r)$, denote $r|_A$ the restriction of the partial function r onto the domain $ann(A)$.

A realisation on annotations r can be applied to an annotated formula A such that $\mathrm{dom}(r) \subseteq \mathrm{ann}(A)$ inductively:

- $\bot^r := \bot$
- $p^r := p$
- $(A * B)^r := (A^r * B^r)$ where $* \in \{\wedge, \vee, \to\}$
- $(\Box_n A)^r := r(n){:}A^r$
- $(\Diamond_n A)^r := r(n){:}A^r$ with the *satisfier self-referentiality restriction* that: if $n = 4k + 3$, α_k does not occur in A^r. $(*)$

By first annotating properly all modalities occurring in it and then replacing each of them by the term $r(n)$ given by the realisation on annotations, we henceforth obtain a realisation A^r for a formula $A \in \mathcal{L}_\Box$, which is in particular *normal* and *restricted* (in the sense of the $(*)$ condition above).

Indeed, when a properly annotated formula $A \in \mathcal{L}_\Box^{\mathsf{ann}}$ is realised through this algorithm, it fixes automatically some of the variables that occur in A^r. Namely, $\{x_n \in \mathsf{PrfVar} \mid \Box_{4n+1} \text{ occurs in } A\} \cup \{\alpha_n \in \mathsf{SatVar} \mid \Diamond_{4n+3} \text{ occurs in } A\}$ designates a set of fixed variables, which we will call $\mathrm{negvar}(A)$.

The satisfier self-referentiality restriction is required when handling realisations over annotated formula. It will become apparent in the proof of Lemma 29.

3 Nested Sequent Calculus

A (full nested) sequent Γ is comprised of two distinct parts: an LHS sequent Λ and an RHS sequent Π. Formally, these sequents are unordered multisets generated from the following grammar:

$$\Lambda ::= \varnothing \mid A^\bullet \mid \langle \Lambda \rangle \mid \Lambda, \Lambda \qquad \Pi ::= A^\circ \mid [\Gamma] \qquad \Gamma ::= \Lambda, \Pi$$

where A ranges over $\mathcal{L}_\Box$. The dots can be seen as marking the polarity of a formula in a sequent: $\bullet$ represents the antecedent; $\circ$ represents the succedent – hence, the restriction of only one $\circ$-formula in a sequent Π is a rendering of Gentzen's intuitionistic restriction [45].

The formula interpretation of nested sequents is defined as follows:

$$\begin{array}{rlrl} \mathrm{form}(\varnothing) &:= \top & & \\ \mathrm{form}(A^\bullet) &:= A & \mathrm{form}(A^\circ) &:= A \\ \mathrm{form}(\langle \Lambda \rangle) &:= \Diamond \mathrm{form}(\Lambda) & \mathrm{form}([\Gamma]) &:= \Box \mathrm{form}(\Gamma) \\ \mathrm{form}(\Lambda_1, \Lambda_2) &:= \mathrm{form}(\Lambda_1) \wedge \mathrm{form}(\Lambda_2) & \mathrm{form}(\Lambda, \Pi) &:= \mathrm{form}(\Lambda) \to \mathrm{form}(\Pi) \end{array}$$

Definition 8 (Contexts). *A* context $\Delta\{\,\}$ *is like a sequent but contains a hole wherever a formula may otherwise occur. We formally distinguish between* output context $\Lambda\{\,\}$ *and* input context $\Gamma\{\,\}$*:*

$$\Lambda\{\,\} ::= \{\,\} \mid \Lambda, \Lambda\{\,\} \mid \langle \Lambda\{\,\} \rangle \qquad \Gamma\{\,\} ::= \Lambda\{\,\}, \Pi \mid \Lambda, \Gamma\{\,\} \mid [\Gamma\{\,\}]$$

Remark 9. An *input context* $\Gamma\{\,\}$ contains an $\circ$-formula, so the hole only needs to be filled with a LHS sequent to give a sequent.
For example, if $\Gamma\{\,\} = \Lambda_1, [\langle \Lambda_2 \rangle, \{\,\}, A^\circ]$, then $\Gamma\{\Lambda_3\} = \Lambda_1, [\langle \Lambda_2 \rangle, \Lambda_3, A^\circ]$.

An *output context* $\Lambda\{\,\}$ on the other hand does not contain an $\circ$-formula, so if the hole is filled with a LHS sequent one simply gets a LHS sequent.
For example, if $\Lambda\{\,\} = \Lambda_1, \langle \langle \Lambda_2 \rangle, \{\,\} \rangle$, then $\Lambda\{\Lambda_3\} = \Lambda_1, \langle \langle \Lambda_2 \rangle, \Lambda_3 \rangle$.

To obtain a sequent, the hole in $\Lambda\{\,\}$ needs to be filled with a sequent too. Furthermore, when doing so, note the need to flip some brackets from $\langle\cdot\rangle$ to $[\cdot]$ (along the path from the root to the hole).
For example, if $\Lambda\{\,\} = \Lambda_1, \langle \langle \Lambda_2 \rangle, \{\,\} \rangle$, then $\Lambda\{\Lambda_3, A^\circ\} = \Lambda_1, [\langle \Lambda_2 \rangle, \Lambda_3, A^\circ]$.

Definition 10 (Depth). *The depth of a context is defined inductively:*

$$\begin{aligned} \mathrm{depth}(\{\,\}) &:= 0 & \mathrm{depth}(\Lambda\{\,\}, \Pi) &:= \mathrm{depth}(\Lambda\{\,\}) \\ \mathrm{depth}(\Lambda_1, \Lambda_2\{\,\}) &:= \mathrm{depth}(\Lambda_2\{\,\}) & \mathrm{depth}(\Lambda, \Gamma\{\,\}) &:= \mathrm{depth}(\Gamma\{\,\}) \\ \mathrm{depth}(\langle \Lambda\{\,\} \rangle) &:= \mathrm{depth}(\Lambda\{\,\}) + 1 & \mathrm{depth}([\Gamma\{\,\}]) &:= \mathrm{depth}(\Gamma\{\,\}) + 1 \end{aligned}$$

Definition 11. *For every input context $\Gamma\{\,\}$, its output pruning $\Gamma^{\downarrow}\{\,\}$ is an output context with the hole in the same position as $\Gamma\{\,\}$ but with the output formula in $\Gamma\{\,\}$ removed and brackets changed from $[\cdot]$ to $\langle\cdot\rangle$ as necessary. Formally,*

$$(\Lambda\{\,\}, \Pi)^{\downarrow} := \Lambda\{\,\}, \Pi^{\downarrow} \qquad (\Lambda, \Gamma\{\,\})^{\downarrow} := \Lambda, \Gamma^{\downarrow}\{\,\} \qquad ([\Gamma\{\,\}])^{\downarrow} := \langle \Gamma^{\downarrow}\{\,\} \rangle$$

where $\Pi^{\downarrow}$ is given inductively by $(A^\circ)^{\downarrow} := \varnothing$ and $([\Lambda, \Pi])^{\downarrow} := \langle \Lambda, \Pi^{\downarrow} \rangle$.

For example, if $\Gamma\{\,\} = \Lambda_1, [\langle \Lambda_2 \rangle, \{\,\}, A^\circ]$, then $\Gamma^{\downarrow}\{\,\} = \Lambda_1, \langle \langle \Lambda_2 \rangle, \{\,\} \rangle$.

The proof system nIK consists of the rules given in Fig. 3. The addition of the rules in Fig. 4 yields the proof systems $\mathsf{nL} \in \{\mathsf{nIK}, \mathsf{nIT}, \mathsf{nIK4}, \mathsf{nIS4}\}$ by picking and mixing rules corresponding to axioms t and $\mathsf{4}$.

A proof of a nested sequent Γ in nL is constructed as trees from these rules with root Γ and leaves closed by axiomatic rules; we then write $\mathsf{nL} \vdash \Gamma$.

Theorem 12 (Straßburger [45]). *Let $\mathsf{L} \in \{\mathsf{IK}, \mathsf{IT}, \mathsf{IK4}, \mathsf{IS4}\}$. For $A \in \mathcal{L}_\Box$:*

$$\mathsf{L} \vdash A \iff \mathsf{nL} \vdash A^\circ$$

The completeness is proved via a cut-elimination argument where the cut rule is of the shape

$$\mathsf{cut}\ \frac{\Gamma^{\downarrow}\{A^\circ\} \quad \Gamma\{A^\bullet\}}{\Gamma\{\varnothing\}}$$

Cut-freeness is in fact a critical property to use a proof system for a syntactic realisation result, see [6, Chapter 8.8] for details.

Another key detail concerns the use of an explicit contraction rule, which the system in [45] does not use, as it is admissible due to the shape of the rules. For the purposes of our work, we require to decouple contraction from the rules similarly to the system in [35].

$$\bot^\bullet \frac{}{\Gamma\{\bot^\bullet\}} \qquad \mathsf{id}\, \frac{}{\Lambda\{p^\bullet, p^\circ\}}$$

$$\mathsf{c}^\bullet \frac{\Gamma\{\Lambda, \Lambda\}}{\Gamma\{\Lambda\}} \qquad \wedge^\bullet \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \qquad \wedge^\circ \frac{\Lambda\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Lambda\{A \wedge B^\circ\}}$$

$$\vee^\bullet \frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \qquad \vee_1^\circ \frac{\Lambda\{A^\circ\}}{\Lambda\{A \vee B^\circ\}} \qquad \vee_2^\circ \frac{\Lambda\{B^\circ\}}{\Lambda\{A \vee B^\circ\}}$$

$$\to^\bullet \frac{\Gamma^\downarrow\{A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \to B^\bullet\}} \qquad \to^\circ \frac{\Lambda\{A^\bullet, B^\circ\}}{\Lambda\{A \to B^\circ\}}$$

$$\Box^\bullet_{[\cdot]} \frac{\Lambda\{[A^\bullet, \Gamma]\}}{\Lambda\{\Box A^\bullet, [\Gamma]\}} \qquad \Box^\bullet_{\langle\cdot\rangle} \frac{\Gamma\{\langle A^\bullet, \Lambda\rangle\}}{\Gamma\{\Box A^\bullet, \langle\Lambda\rangle\}} \qquad \Box^\circ \frac{\Lambda\{[A^\circ]\}}{\Lambda\{\Box A^\circ\}}$$

$$\Diamond^\bullet \frac{\Gamma\{\langle A^\bullet\rangle\}}{\Gamma\{\Diamond A^\bullet\}} \qquad \Diamond^\circ \frac{\Lambda_1\{[A^\circ, \Lambda_2]\}}{\Lambda_1\{\Diamond A^\circ, \langle\Lambda_2\rangle\}}$$

Fig. 3. System nIK

$$\mathsf{t}^\bullet \frac{\Gamma\{A^\bullet\}}{\Gamma\{\Box A^\bullet\}} \qquad \mathsf{t}^\circ \frac{\Lambda\{A^\circ\}}{\Lambda\{\Diamond A^\circ\}}$$

$$\mathsf{4}^\bullet_{[\cdot]} \frac{\Lambda\{[\Box A^\bullet, \Gamma]\}}{\Lambda\{\Box A^\bullet, [\Gamma]\}} \qquad \mathsf{4}^\bullet_{\langle\cdot\rangle} \frac{\Gamma\{\langle\Box A^\bullet, \Lambda\rangle\}}{\Gamma\{\Box A^\bullet, \langle\Lambda\rangle\}} \qquad \mathsf{4}^\circ \frac{\Lambda_1\{[\Diamond A^\circ, \Lambda_2]\}}{\Lambda_1\{\Diamond A^\circ, \langle\Lambda_2\rangle\}}$$

Fig. 4. Modal rules for t and $\mathsf{4}$

Example 13. This is an example of a proof in nIK:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bot^\bullet \dfrac{}{\bot^\circ, \langle\bot^\bullet\rangle}}{\Box^\bullet \dfrac{}{\Box\bot^\bullet, \bot^\circ, \langle\varnothing\rangle}}}{\to^\circ \dfrac{}{\Box\bot \to \bot^\circ, \langle\varnothing\rangle}} \quad \bot^\bullet \dfrac{}{\bot^\bullet, [\bot^\circ]}}{\to^\bullet \dfrac{}{(\Box\bot \to \bot) \to \bot^\bullet, [\bot^\circ]}}}{\Box^\circ \dfrac{}{((\Box\bot \to \bot) \to \bot)^\bullet, \Box\bot^\circ}}}{\to^\circ \dfrac{}{((\Box\bot \to \bot) \to \bot) \to \Box\bot^\circ}}$$

Note that this is a theorem of IK but not of CK [17]. We will expand this example later to illustrate how our system allows us to provide a realisation of it.

3.1 Annotated Sequents

Definition 14 (Annotated sequents). *An* annotated sequent/context *is a sequent/context in which only annotated formulas occur, and brackets* $[\cdot]$ *and* $\langle\cdot\rangle$ *are indexed by some* $n \in \mathbb{N}$*, written* $[\cdot]_n$ *and* $\langle\cdot\rangle_n$*.*

$$\mathrm{form}(\langle\Lambda\rangle_n) := \Diamond_n \mathrm{form}(\Lambda) \qquad \mathrm{form}([\Gamma]_n) := \Box_n \mathrm{form}(\Gamma)$$

Most definitions for annotated formulas and for nested sequents transfer directly to annotated sequents. However, the concept of output pruning requires some careful handling as not only the brackets need to be possibly flipped, but their annotations need to be accordingly and appropriately changed as well.

For example, if $\Gamma\{\,\} = \langle \Lambda_1, \{\,\}\rangle_{4k+3}, [\Lambda_2, C^\circ]_{4n}$, then for some $m \in \mathbb{N}$ and a re-annotated version of Λ_2 given as Λ_2', we define $\Gamma^{\downarrow}\{\,\} = \langle \Lambda_1, \{\,\}\rangle_{4k+3}, \langle \Lambda_2'\rangle_{4m+3}$. Finally, $\Gamma^{\downarrow}\{A^\circ\} = [\Lambda_1, A^\circ]_{4l}, \langle \Lambda_2'\rangle_{4m+3}$ for some $l \in \mathbb{N}$. This is necessary to ensure square brackets are annotated by even numbers and angle brackets by odd numbers.

The annotated rules are also mainly the same as the ones without annotations. The contraction rule, however, applies in principle to the same sequent, but needs to distinguish annotations, that is, it should be written:

$$\mathsf{c}^\bullet \frac{\Gamma\{\Lambda_1, \Lambda_2\}}{\Gamma\{\Lambda_3\}}$$

where Λ_1, Λ_2 and Λ_3 represent the same unannotated sequent but do not share annotation indices.

The subtlety of the annotated output pruning is key to the $\to^\bullet$-rule.

Example 15. This is illustrated on the following instance of $\to^\bullet$:

$$\to^\bullet \frac{[\Lambda_1, A^\circ]_{4l}, \langle \Lambda_2'\rangle_{4m+3} \qquad \langle \Lambda_1, B^\bullet\rangle_{4k+3}, [\Lambda_2'', C^\circ]_{4n}}{\langle \Lambda_1, A \to B^\bullet\rangle_{4k+3}, [\Lambda_2, C^\circ]_{4n}}$$

where $\Lambda_2, \Lambda_2', \Lambda_2''$ represent the same unannotated sequent but do not share annotations.

From now on, a nested sequent system nL will casually refer to the annotated version. We note that Theorem 12 also holds for the annotated case – for any unannotated derivation in the nested sequent system, properly annotate the endsequent and propagate the annotations upwards.

Example 16. Proof from Example 13 in an annotated version:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\bot^\bullet}}{\bot^\circ, \langle \bot^\bullet\rangle_3}\Box^\bullet_{\langle\cdot\rangle}}{\Box_1\bot^\bullet, \bot^\circ, \langle\varnothing\rangle_3}\to^\circ}{\Box_1\bot \to \bot^\circ, \langle\varnothing\rangle_3} \quad \dfrac{\dfrac{}{\bot^\bullet}\bot^\bullet}{\bot^\bullet, [\bot^\circ]_0}}{(\Box_1\bot \to \bot) \to \bot^\bullet, [\bot^\circ]_0}\to^\bullet}{((\Box_1\bot \to \bot) \to \bot)^\bullet, \Box_0\bot^\circ}\Box^\circ}{((\Box_1\bot \to \bot) \to \bot) \to \Box_0\bot^\circ}\to^\circ$$

We observe moreover that the soundness of the $\to^\bullet$ rule combines modal reasoning for the $\mathsf{k_4}$ axiom and propositional reasoning for implication. It is

helpful in the realisation proof to decouple these two aspects and replace each occurrence of $\to^\bullet$ in a proof with a macro rule composed of $\to^\bullet_{\mathsf{s}}$, $\triangleright$ and $\mathsf{c}^\bullet$ rules:

$$\to^\bullet_{\mathsf{s}} \frac{\Lambda_1\{\Lambda_2\{A^\circ\}, \Lambda\} \quad \Lambda_1\{\Lambda_2\{B^\bullet\}, \Pi\}}{\Lambda_1\{\Lambda_2\{A \to B^\bullet\}, \Lambda, \Pi\}} \qquad \triangleright \frac{\Gamma\{\langle \Lambda_1 \rangle_{4k+3}, [\Lambda_2, \Pi]_{4n}\}}{\Gamma\{[\Lambda_1, \Lambda_2, \Pi]_{4n}\}}$$

where Λ_1 and Λ_2 are annotated LHS sequents and Π is an annotated RHS sequent. The soundness of the $\to^\bullet_{\mathsf{s}}$ rule only makes use of basic modal reasoning and propositional reasoning for implication and the soundness of the $\triangleright$ rule makes use of the $\mathsf{k_4}$ axiom. For a detailed proof of how an occurrence of the $\to^\bullet$ rule is replaced, the reader is referred to [34].

Example 17. We rewrite the annotated rule instance of Example 15 with these macro rules:

$$\to^\bullet_{\mathsf{s}} \frac{[\Lambda_1, A^\circ]_{4l}, \langle \Lambda_2' \rangle_{4m+3} \quad \langle \Lambda_1, B^\bullet \rangle_{4k+3}, [\Lambda_2'', C^\circ]_{4n}}{\triangleright \dfrac{\langle \Lambda_1, A \to B^\bullet \rangle_{4k+3}, \langle \Lambda_2' \rangle_{4m+3}, [\Lambda_2'', C^\circ]_{4n}}{\mathsf{c}^\bullet \dfrac{\langle \Lambda_1, A \to B^\bullet \rangle_{4k+3}, [\Lambda_2', \Lambda_2'', C^\circ]_{4n}}{\langle \Lambda_1, A \to B^\bullet \rangle_{4k+3}, [\Lambda_2, C^\circ]_{4n}}}}$$

where the contraction rule $\mathsf{c}^\bullet$ applies to the LHS sequents Λ_2', Λ_2'' to obtain a differently annotated sequent Λ_2 but all representing the same sequent. Reading the derivation bottom-up, we note that the annotations of brackets on the left premiss of the $\to^\bullet_{\mathsf{s}}$ are changed to accommodate the change of bracket type.

4 Realisation Theorem

A realisation can be extended to an annotated sequent Σ as $\Sigma^r := \mathrm{form}(\Sigma)^r$.

The goal of this section is to prove the following theorem.

Theorem 18 (Realisation Theorem). *Let* $\mathsf{L} \in \{\mathsf{IK}, \mathsf{IT}, \mathsf{IK4}, \mathsf{IS4}\}$. *Let* Γ *be an annotated sequent. If*

$$\mathsf{nL} \vdash \Gamma$$

then there exists a realisation function r *on* Γ *such that*

$$\mathsf{JL} \vdash \Gamma^r$$

The high-level idea is to systematically scan through the proof of Γ top down. In the leaves, a basic realisation is given to any axiomatic sequent Σ defined as:

for any n an annotation in Σ, $r(n) := \begin{cases} y_m & \text{if} \quad n = 4m \\ x_m & \text{if} \quad n = 4m+1 \\ \beta_m & \text{if} \quad n = 4m+2 \\ \alpha_m & \text{if} \quad n = 4m+3 \end{cases}$

Example 19. On the annotated proof from Example 16, at each stage of the proof, the realisation procedure will yields a justification formula for each sequent in the derivation tree. These formulas can be linked by derivations in the

Hilbert system of the justification logics. The variables y_i and β_j are step-by-step substituted top-down by larger and larger terms according to the JL axioms.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha_0{:}\bot \to \bot}\,\bot^\bullet}{(\alpha_0{:}\top \wedge x_0{:}\bot) \to \bot}\,\Box^\bullet}{\alpha_0{:}\top \to (x_0{:}\bot \to \bot)}\,\to^\circ \qquad \dfrac{}{\bot \to y_0{:}\bot}\,\bot^\bullet}{((x_0{:}\bot \to \bot) \to \bot) \to t(\alpha_0 \triangleright y_0){:}\bot}\,\to^\bullet}{((x_0{:}\bot \to \bot) \to \bot) \to t(\alpha_0 \triangleright y_0){:}\bot}\,\Box^\circ}{((x_0{:}\bot \to \bot) \to \bot) \to t(\alpha_0 \triangleright y_0){:}\bot}\,\to^\circ$$

The proof term $t(\alpha_k \triangleright y_m)$ will be constructed from the steps given in Lemma 26. Using the $\mathsf{jk_4}$ axiom

$$\mathsf{JL} \vdash (\alpha_k{:}\top \to y_m{:}\bot) \to \alpha_k \triangleright y_m{:}(\top \to \bot)$$

and using the Lifting Lemma 21 on the propositional theorem

$$\mathsf{JL} \vdash (\top \to \bot) \to \bot$$

there exists a proof term $t(\alpha_k \triangleright y_m)$ such that

$$\mathsf{JL} \vdash \alpha_k \triangleright y_m{:}(\top \to \bot) \to t(\alpha_k \triangleright y_m){:}\bot$$

In the following subsections we will provide some of the key lemmas that allow us to eventually complete the realisation proof. They all apply for any $\mathsf{JL} \in \{\mathsf{JIK}, \mathsf{JIT}, \mathsf{JIK4}, \mathsf{JIS4}\}$.

4.1 Lifting

Lemma 20 (Internalised neccessitation). *Let $A \in \mathcal{L}_\mathsf{J}$. If $\mathsf{JL} \vdash A$, then there exists a ground term t such that $\mathsf{JL} \vdash t{:}A$.*

Proof. The proof is the same as the classical case. Example proofs can be found in [6,33]. □

Lemma 21 (Lifting Lemma). *Let $B_1, \ldots, B_n$, $C, A \in \mathcal{L}_\mathsf{J}$ for some $n \in \mathbb{N}$.*

1. *If $\mathsf{JL} \vdash B_1 \to B_2 \to \cdots \to B_n \to A$, then for proof terms $s_1, \ldots, s_n$, there exists a proof term $t(s_1, \ldots, s_n)$ such that*

$$\mathsf{JL} \vdash s_1{:}B_1 \to \cdots \to s_n{:}B_n \to t(s_1, \ldots, s_n){:}A$$

2. *If $\mathsf{JL} \vdash B_1 \to B_2 \to \cdots \to B_n \to C \to A$, then for proof terms $s_1, \ldots, s_n$ and satisfier ν, there exists a satisfier $\mu(s_1, \ldots, s_n, \nu)$ such that*

$$\mathsf{JL} \vdash s_1{:}B_1 \to \cdots \to s_n{:}B_n \to \nu{:}C \to \mu(s_1, \ldots, s_n, \nu){:}A$$

Proof. A proof of 1. can be found in [6,33]. A proof of 2. can be found in [30]. The proof is not sensitive to the additional axioms and operators used here. □

4.2 Merging

Definition 22 (Substitution). *A* substitution *is a map* $\sigma : \mathsf{PrfVar} \to \mathsf{PrfTm}$ *and* $\sigma : \mathsf{SatVar} \to \mathsf{SatTm}$. *Substitutions can then be inductively extended to* PrfTm, SatTm *and* $\mathcal{L}_{\mathsf{J}}$ *as standard. We will write* $\chi\sigma$ *for* $\sigma(\chi)$ *for some* $\chi \in \mathsf{PrfTm} \cup \mathsf{SatTm} \cup \mathcal{L}_{\mathsf{J}}$.

The domain *of* σ *is the set* $dom(\sigma) := \{\chi \in \mathsf{PrfVar} \cup \mathsf{SatVar} \mid \chi\sigma \neq \chi\}$.

Lemma 23 (Substitution Lemma). *Let* $A \in \mathcal{L}_{\mathsf{J}}$. *Let* σ *be a substitution. If* $\mathsf{JL} \vdash A$, *then* $\mathsf{JL} \vdash A\sigma$.

Proof. This is a routine proof which follows precisely from the fact that given any axiom instance A, $A\sigma$ is an axiom instance, and hence, for a conclusion of the can rule $c_n{:}\ldots{:}c_1{:}A$, the result of applying a substitution on it: $(c_n{:}\ldots{:}c_1{:}A)\sigma = c_n{:}\ldots{:}c_1{:}(A\sigma)$ is a conclusion of the can rule. □

Theorem 24 (Realisation merging). *Let* $A \in \mathcal{L}^{\mathsf{ann}}_{\Box}$. *Let* r_1 *and* r_2 *be realisations on* A. *Then there exists a realisation function* r *on* A *and a substitution* σ *such that:*

1. *For every positive subformula* X *of* A, $\mathsf{JL} \vdash X^{r_1}\sigma \to X^{r_2}\sigma \to X^r$
2. *For every negative subformula* X *of* A, $\mathsf{JL} \vdash X^r \to X^{r_i}\sigma$ *where* $i \in \{1, 2\}$.

where $dom(\sigma) \subseteq negvar(A)$ *and* $x\sigma$ *contains no new variables for each* x.

Proof. This is an adaptation of Fitting [23,24]. The original proof is in the setting of classical Logic of Proofs but does not make use of classical reasoning or the $\mathsf{jt}_{\Box}$ and $\mathsf{j4}_{\Box}$ axioms. The proof can be adapted to deal with satisfiers and has a similar treatment to proof terms, with $\sqcup$ playing for satisfiers the role played by $+$ for proof terms. □

These notions are similarly extended to annotated sequents.

Corollary 25. *Let* Γ *be an annotated sequent and* Λ *an annotated LHS sequent. Let* r_1 *and* r_2 *be realisation functions. Then there exists a realisation function* r *and a substitution* σ *such that:*

1. $\mathsf{JL} \vdash \Gamma^{r_1}\sigma \to \Gamma^{r_2}\sigma \to \Gamma^r$ *where* $dom(\sigma) \subseteq negvar(\Gamma)$ *and* $x\sigma$ *contains no new variables*
2. $\mathsf{JL} \vdash \Lambda^r \to \Lambda^{r_i}\sigma$ *where* $i \in \{1, 2\}$, $dom(\sigma) \subseteq negvar(\Gamma)$ *and* $x\sigma$ *contains no new variables.*

4.3 Proving Realisation

Given an instance of a $\mathsf{rule} \in \{\mathsf{id}, \wedge^\circ, \vee^\circ, \to^\circ, \Box^\circ, \Diamond^\circ, \Box^\bullet_{[\cdot]}, \mathsf{t}^\circ, 4^\circ, 4^\bullet_{[\cdot]}, \triangleright\}$, for some $k \leq 2$, it will be of the shape:

$$\mathsf{rule}\ \frac{\Lambda\{\Gamma_1\} \quad \cdots \quad \Lambda\{\Gamma_k\}}{\Lambda\{\Gamma_0\}}$$

As these rules behave similarly to classical rules, following [25] we can show:

Lemma 26. *For any realisations r_i on $\Lambda\{\Gamma_i\}$ for $i \leq k$, there exist a realisation r on $\Lambda\{\Gamma_0\}$ and substitutions σ_i for $i \leq k$, such that*

$$\mathsf{JL} \vdash (\Lambda\{\Gamma_1\}^{r_1}\sigma_1 \wedge \cdots \wedge \Lambda\{\Gamma_k\}^{r_k}\sigma_k) \to \Lambda\{\Gamma_0\}^{r}$$

Proof (Sketch). We proceed by induction on $\Lambda\{\,\}$. See [34] for the full details.

The base case, $\Lambda = \varnothing$, relies on a bespoke proof for each specific rule. These are similar to the classical case for id, $\vee^\circ$, $\wedge^\circ$ and $\to^\circ$. Note the particular case of id where the empty conjunction reduces and the particular case of $\wedge^\circ$ whose branching requires the merging property (Theorem 24). However, rules that involve a $\Diamond$-formula (or $\langle\cdot\rangle$-brackets) require a novel treatment as they should introduce satisfier terms, rather than the classical treatment through duality with $\Box$ which would only introduce proof terms.

The inductive cases, $\Lambda = A^\bullet, \Lambda'$ and $\Lambda = \langle\Lambda'\rangle$, follow the shallow-to-deep approach first devised in [25] but using intuitionistic reasoning and applying lifting (Lemma 21) for both proof and satisfier terms. □

Given an instance of a $\mathsf{rule} \in \{\bot^\bullet, \vee^\bullet, \wedge^\bullet, \Box^\bullet, \Diamond^\bullet, \to^\bullet_{\mathsf{s}}, \mathsf{c}^\bullet, \mathsf{t}^\bullet, \mathsf{4}^\bullet\}$, for some $k \leq 2$, it will be of the shape:

$$\mathsf{rule}\,\frac{\Gamma\{\Lambda_1\} \quad \cdots \quad \Gamma\{\Lambda_k\}}{\Gamma\{\Lambda_0\}}$$

This can be further refined depending of the precise position of the Λ_i within context $\Gamma\{\,\}$. If the principal formula is among the LHS part of $\Gamma\{\,\}$, the rule can be rewritten:

$$\mathsf{rule}\,\frac{\Lambda\{\Lambda_1\}, \Pi \quad \cdots \quad \Lambda\{\Lambda_k\}, \Pi}{\Lambda\{\Lambda_0\}, \Pi}$$

In this case, we get the following upside-down reading of the previous lemma:

Lemma 27. *For any realisations r_i on $\Lambda\{\Lambda_i\}$ for $i \leq k$, there exist a realisation r on $\Lambda\{\Lambda_0\}$ and substitutions σ_i for $i \leq k$, such that*

$$\mathsf{JL} \vdash \Lambda\{\Lambda_0\}^{r} \to (\Lambda\{\Lambda_1\}^{r_1}\sigma_1 \vee \cdots \vee \Lambda\{\Lambda_k\}^{r_k}\sigma_k)$$

Proof (Sketch). As before, this proceeds by induction on $\Lambda\{\,\}$. See [34] for details. Again, in the case of the axiomatic $\bot^\bullet$, the empty disjunction reduces to $\bot$, and in the case of the branching $\vee^\bullet$, merging (Theorem 24) is required to reconcile the realisations of the branches. The details for $\vee^\bullet$ are given as Lemma 29 in Appendix; it also illustrate the need for the *diamond self-referential restriction*. □

On the other hand, if the principal formula is among the RHS of $\Gamma\{\,\}$, rule presents itself as

$$\mathsf{rule}\,\frac{\Lambda, \Pi\{\Lambda_1\} \quad \cdots \quad \Lambda, \Pi\{\Lambda_k\}}{\Lambda, \Pi\{\Lambda_0\}}$$

which lets us prove a similar lemma to the classical case and Lemma 26:

Lemma 28. *For any realisations r_i on $\Lambda\{\Gamma_i\}$ for $i \leq k$, there exist a realisation r on $\Lambda\{\Gamma_0\}$ and substitutions σ_i for $i \leq k$, such that*

$$\mathsf{JL} \vdash (\Pi\{\Lambda_1\}^{r_1}\sigma_1 \wedge \cdots \wedge \Pi\{\Lambda_k\}^{r_k}\sigma_k) \to \Pi\{\Lambda_0\}^{r}$$

Proof (Sketch). The proof proceeds by induction on $\Pi\{\,\}$. See all details in [34]. The base case, when $\Pi\{\,\} = A^\circ, \{\,\}$ for some $A \in \mathcal{L}_\Box^{\mathsf{ann}}$, depends on the logical content of each rule. The inductive case, when $\Pi = [\Gamma\{\,\}]$, is where the complexity resides. Indeed, it presents the same dichotomy as described above, where the principal formula can occur in the LHS or the RHS of $\Gamma\{\,\}$. In the latter, the inductive hypothesis applies readily, while in the former, we need Lemma 27 to conclude. The details for $\vee^\bullet$ are given as Lemma 30 in Appendix. □

Finally, putting all the ingredients together,

Proof (Proof of Theorem 18). We proceed by induction on the structure of the proof π of Γ in nL. For the base case, Γ is a conclusion of the id or $\bot^\bullet$ rule. For the inductive case, Γ is the conclusion of the following rule rule

$$\mathsf{rule}\ \dfrac{\overset{\pi_1}{\Gamma_1} \quad \cdots \quad \overset{\pi_i}{\Gamma_i}}{\Gamma}$$

for smaller proofs $\pi_1, \ldots, \pi_k$ and premisses $\Gamma_1, \ldots, \Gamma_k$ where $i \in \{1, \ldots, k\}$. By the inductive hypothesis, we have realisations r_i on Γ_i such that

$$\mathsf{JL} \vdash {\Gamma_i}^{r_i}$$

Applying Lemma 26 or 27 corresponding to the rule rule, there exists a realisation r on Γ and substitutions $\sigma_1, \ldots, \sigma_i$ (which can be Id if not mentioned in the Lemmas) such that

$$\mathsf{JL} \vdash ({\Gamma_1}^{r_1}\sigma_1 \wedge \cdots \wedge {\Gamma_i}^{r_i}\sigma_i) \to \Gamma^r$$

or equivalently

$$\mathsf{JL} \vdash {\Gamma_1}^{r_1}\sigma_1 \to \cdots \to {\Gamma_i}^{r_i}\sigma_i \to \Gamma^r$$

By the Substitution Lemma 23

$$\mathsf{JL} \vdash {\Gamma_j}^{r_j}\sigma_j$$

for each $j \in \{1, \ldots, i\}$. Using modus ponens, we achieve

$$\mathsf{JL} \vdash \Gamma^r$$

□

We thus obtain the realisation theorem as a corollary:

Proof (Proof of Theorem 4). By Theorem 12, there is a nested sequent derivation of A° and we apply Theorem 18 to construct a realisation. □

5 Conclusion and Future Work

We have presented a justification counterpart for intuitionistic modal logic, with a modified merging theorem for realisations and a method for a realisation theorem using intuitionistic nested sequents. This method should be adaptable to other cut-free nested sequent systems, e.g., for the IS5 cube [45,35], and to some logics of the CS5 cube [1] – the challenge here being finding suitable justification counterparts for other modal axioms of the cube.

The restriction of only one output formula in the nested sequent system can be removed to recover a proof system for the modal logics over a classical base [31]. The method of realisation presented in this work can also be adapted to this system. It would provide an alternative realisation result for classical modal logics, this time with both $\Box$ and $\Diamond$ being made explicit. Given that IK and its extensions satisfy Simpson's requirements [43], namely the addition of the law of excluded middle recovers the corresponding classical modal logic, this could provide an axiomatic direction to understanding a classical version of satisfiers.

Justification logic for classical modal logic can be understood semantically over various classes of models [38,22,5,32]. Some of these have been adapted to a basic constructive setting, with no $\Diamond$-modalities, in [36] using birelational models, which are standardly used for intuitionistic/constructive modal logics [43]. These could therefore provide a starting point to understand justification logics for CK [30] and for IK (this work) semantically. There could furthermore be a way to understand the $\mathsf{jk}_3, \mathsf{jk}_4, \mathsf{jk}_5$ axioms modularly similar to the case in intuitionistic modal logic [26].

Logic of Proofs LP, i.e., the justification logic for classical modal logic S4, is in particular complete with respect to an arithmetic interpretation into PA [2]. Previous investigations towards justification logics interpretable into HA were performed without the $\Diamond$-modality and required heavy machinery to recover the full power of HA. We know however that by adding $\Diamond$-modalities we at least regain access to the Gödel-Gentzen (double-negation) translation. It would be interesting to compare the strength of the justification logic we obtained here for IS4 wrt. arithmetical completeness, and whether we can make progress towards an intuitionitic logic of proofs ILP.

A Appendix

We illustrate the technicalities required in the proof of the realisation theorem on the case of the $\vee^\bullet$ rule.

Lemma 29 (Lemma 27 for $\vee^\bullet$). *Given realisation functions r_1 on $\Lambda\{A^\bullet\}$ and r_2 on $\Lambda\{B^\bullet\}$, there exist a substitution σ with $dom(\sigma) \subseteq negvar(\Lambda\{A \vee B^\bullet\})$ and a realisation function r on $\Lambda\{A \vee B^\bullet\}$ such that*

$$\mathsf{JL} \vdash \Lambda\{A \vee B^\bullet\}^r \to (\Lambda\{A^\bullet\}^{r_1}\sigma \vee \Lambda\{B^\bullet\}^{r_2}\sigma)$$

Proof. We proceed by induction on the depth of $\Lambda\{\,\}$.

For the base case, $\Lambda\{\,\} = \Lambda_0, \{\,\}$ for some annotated LHS sequent Λ_0. Using Theorem 24 and Corollary 25 there exist a realisation r and substitution σ such that ${\Lambda_0}^r \to {\Lambda_0}^{r_i}\sigma$, $A^r \to A^{r_1}\sigma$ and $B^r \to B^{r_2}\sigma$ are theorems of JL. It follows by propositional reasoning that

$$\mathsf{JL} \vdash (\Lambda_0, A \vee B^\bullet)^r \to ((\Lambda_0, A^\bullet)^{r_1}\sigma \vee (\Lambda_0, B^\bullet)^{r_2}\sigma)$$

since $(\Lambda_0, A \vee B^\bullet)^r = ({\Lambda_0}^r \wedge (A^r \vee B^r))$ and $(\Lambda_0, A^\bullet)^{r_i}\sigma = ({\Lambda_0}^{r_i}\sigma \wedge A^{r_i}\sigma)$ for each $i \in \{1, 2\}$.

For the inductive case, $\Lambda\{\,\} = \Lambda_0, \langle \Lambda_1\{\,\}\rangle_{4i+3}$ for some annotated LHS sequent Λ_0. By the inductive hypothesis, there exist a realisation r' on $\Lambda_1\{A \vee B^\bullet\}$ and substitution σ' with $\mathrm{dom}(\sigma') \subseteq \mathrm{negvar}(\Lambda_1\{A \vee B^\bullet\})$ such that

$$\mathsf{JL} \vdash \Lambda_1\{A \vee B^\bullet\}^{r'} \to (\Lambda'\{A^\bullet\}^{r_1}\sigma_1 \vee \Lambda'\{B^\bullet\}^{r_2}\sigma_1)$$

Note that $(\sigma' \circ r_i)|_{\Lambda_0}$ is a realisation on Λ_0 for each $i \in \{1, 2\}$. By Corollary 25, there exist a realisation r'' on Λ_0 and substitution σ'' with $\mathrm{dom}(\sigma'') \subseteq \mathrm{negvar}(\Lambda_0)$ such that

$$\mathsf{JL} \vdash {\Lambda_0}^{r''} \to {\Lambda_0}^{\sigma' \circ r_i}\sigma'' = {\Lambda_0}^{\sigma'' \circ \sigma' \circ r_i}$$

for each $i \in \{1, 2\}$. By the Substitution Lemma 23

$$\mathsf{JL} \vdash \Lambda_1\{A \vee B^\bullet\}^{\sigma'' \circ r'} \to (\Lambda_1\{A^\bullet\}^{r_1}(\sigma'' \circ \sigma') \vee \Lambda_1\{B^\bullet\}^{r_2}(\sigma \circ \sigma'))$$

as $\Lambda_1\{A \vee B^\bullet\}^{\sigma'' \circ r'} = \Lambda_1\{A \vee B^\bullet\}^{r'}\sigma''$. By the Lifting Lemma 21, there exists a satisfier term $\mu(\alpha_k)$ such that

$$\mathsf{JL} \vdash \alpha_k{:}\Lambda_1\{A \vee B^\bullet\}^{\sigma'' \circ r'} \to \mu(\alpha_k){:}(\Lambda_1\{A^\bullet\}^{r_1}(\sigma'' \circ \sigma') \vee \Lambda_1\{B^\bullet\}^{r_2}(\sigma \circ \sigma'))$$

Using the jk_3 axiom and transitivity

$$\mathsf{JL} \vdash \alpha_k{:}\Lambda_1\{A \vee B^\bullet\}^{\sigma'' \circ r'} \to (\mu(\alpha_k){:}\Lambda_1'\{A^\bullet\}^{r_1}(\sigma'' \circ \sigma') \vee \mu(\alpha_k){:}\Lambda_1'\{A^\bullet\}^{r_2}(\sigma'' \circ \sigma'))$$

Set σ''' as the substitution with $\alpha_k \mapsto \mu(\alpha_k)$. With the satisfier self-referential restriction,

$$\langle \Lambda_1\{A^\bullet\}\rangle_{4k+3}{}^{r_1}(\sigma''' \circ \sigma'' \circ \sigma') = \mu(\alpha_k){:}\Lambda_1\{A^\bullet\}^{r_1}(\sigma'' \circ \sigma')$$
$$\langle \Lambda_1\{B^\bullet\}\rangle_{4k+3}{}^{r_2}(\sigma''' \circ \sigma'' \circ \sigma') = \mu(\alpha_k){:}\Lambda_1\{B^\bullet\}^{r_2}(\sigma'' \circ \sigma')$$

The Substitution Lemma 23 gives

$$\mathsf{JL} \vdash {\Lambda_0}^{\sigma''' \circ r''} = {\Lambda_0}^{r''}\sigma''' \to {\Lambda_0}^{\sigma'' \circ \sigma' \circ r_i}\sigma''' = {\Lambda_0}^{\sigma''' \circ \sigma'' \circ \sigma' \circ r_i}$$

Set $\sigma = \sigma''' \circ \sigma'' \circ \sigma'$. By propositional reasoning

$$\mathsf{JL} \vdash ({\Lambda_0}^{\sigma''' \circ r''} \wedge \alpha_k{:}\Lambda_1\{A \vee B^\bullet\}^{\sigma'' \circ r'}) \to ((\Lambda_0, \langle \Lambda_1\{A^\bullet\}\rangle_{4k+3})^{r_1}\sigma \vee (\Lambda_0, \langle \Lambda_1\{B^\bullet\}\rangle_{4k+3})^{r_2}\sigma)$$

since $(({\Lambda_0}^{r_i}(\sigma'' \circ \sigma') \wedge \mu(\alpha_k){:}\Lambda_1\{A^\bullet\}^{r_i}(\sigma'' \circ \sigma')) = (\Lambda_0, \langle \Lambda_1\{A^\bullet\}\rangle_{4k+3})^{r_i}\sigma$ for each $i \in \{1, 2\}$.

Finally set $r := (\sigma''' \circ r'')|_{\Lambda_0} \cup (\sigma'' \circ r')|_{\Lambda_1\{A \vee B^\bullet\}} \cup \{(4k+3, \alpha_k)\}$ which is a realisation function on $\Lambda\{A \vee B^\bullet\}$. □

Lemma 30 (Lemma 28 for $\vee^\bullet$). *Given realisation functions r_1 and r_2 on $\Gamma\{A^\bullet\}$ and $\Gamma\{B^\bullet\}$ respectively, there exist a realisation function r and a substitution σ such that*

$$\mathsf{JL} \vdash \Gamma\{A^\bullet\}^{r_1}\sigma \to \Gamma\{B^\bullet\}^{r_2}\sigma \to \Gamma\{A \vee B^\bullet\}^{r}$$

Proof. We rewrite $\Gamma\{\,\}$ as $\Gamma\{\Lambda\{\,\}, \Pi\}$ and proceed by induction on $\mathrm{depth}(\Gamma\{\,\})$.

For the base case, $\Gamma\{\Lambda\{\,\}, \Pi\} = \Lambda\{\,\}, \Pi$. By Corollary 25, there exist a realisation r' on Π and a substitution σ' with $\mathrm{dom}(\sigma') \subseteq \mathrm{negvar}(\Pi)$ such that

$$\mathsf{JL} \vdash \Pi^{r_1}\sigma' \to \Pi^{r_2}\sigma' \to \Pi^{r'}$$

Note that $(\sigma' \circ r_1)|_{\Lambda\{A^\bullet\}}$ and $(\sigma' \circ r_2)|_{\Lambda\{B^\bullet\}}$ are realisation functions. By Lemma 29, there exist a realisation r'' on $\Lambda\{A \vee B^\bullet\}$ and a substitution σ'' with $\mathrm{dom}(\sigma'') \subseteq \mathrm{negvar}(\Lambda\{A \vee B^\bullet\})$ such that

$$\mathsf{JL} \vdash \Lambda\{A \vee B^\bullet\}^{r''} \to (\Lambda\{A^\bullet\}^{\sigma' \circ r_1}\sigma'' \vee \Lambda\{B^\bullet\}^{\sigma' \circ r_2}\sigma'')$$

By the Substitution Lemma 23,

$$\mathsf{JL} \vdash \Pi^{r_1}(\sigma'' \circ \sigma') \to \Pi^{r_2}(\sigma'' \circ \sigma') \to \Pi^{r'}\sigma''$$

Equivalently

$$\mathsf{JL} \vdash \Pi^{r_1}(\sigma'' \circ \sigma') \to \Pi^{r_2}(\sigma'' \circ \sigma') \to \Pi^{\sigma'' \circ r'}$$

and by propositional reasoning

$$\begin{aligned}\mathsf{JL} \vdash (\Lambda\{A^\bullet\}^{r_2}(\sigma'' \circ \sigma') \to \Pi^{r_1}(\sigma'' \circ \sigma')) \\ \to (\Lambda\{B^\bullet\}^{r_2}(\sigma'' \circ \sigma') \to \Pi^{r_2}(\sigma'' \circ \sigma')) \\ \to (\Lambda\{A \vee B^\bullet\}^{r''} \to \Pi^{\sigma'' \circ r'})\end{aligned}$$

Set $r := r' \cup (\sigma'' \circ r')$ and $\sigma := \sigma'' \circ \sigma'$.

For the inductive case, $\Gamma\{\,\} = \Lambda, [\Gamma'\{\,\}]_{4n}$ for some annotated LHS sequent Λ and context $\Gamma'\{\,\}$. Using the inductive hypothesis, there exists a realisation r' on $\Gamma'\{A \vee B^\bullet\}$ and substitution σ' with $\mathrm{dom}(\sigma) \subseteq \mathrm{negvar}(\Gamma'\{A \vee B^\bullet\})$ such that

$$\mathsf{J} \vdash \Gamma'\{A^\bullet\}^{r_1}\sigma' \to \Gamma'\{B^\bullet\}^{r_2}\sigma' \to \Gamma'\{A \vee B^\bullet\}^{r'}$$

Similarly, $(\sigma' \circ r_i)|_\Lambda$ is a realisation on Λ for each $i \in \{1, 2\}$. By Corollary 25, there exist a realisation r'' on Λ and substitution σ'' with $\mathrm{dom}(\sigma'') \subseteq \mathrm{negvar}(\Lambda)$ such that $\mathsf{J} \vdash \Lambda^{r''} \to \Lambda^{\sigma' \circ r_i}\sigma''$ for $i \in \{1, 2\}$. By the Substitution Lemma 23

$$\mathsf{J} \vdash \Gamma'\{A^\bullet\}^{r_1}(\sigma'' \circ \sigma') \to \Gamma'\{B^\bullet\}^{r_2}(\sigma'' \circ \sigma') \to \Gamma'\{A \vee B^\bullet\}^{r'}\sigma''$$

Set $\sigma = \sigma'' \circ \sigma$. By the Lifting Lemma 21, there exists a proof term $t \equiv t((\sigma \circ r_1)(4n), (\sigma \circ r_2)(4n))$ such that

$$\mathsf{J} \vdash (\sigma \circ r_1)(4n){:}\Gamma'\{A^\bullet\}^{r_1}\sigma \to (\sigma \circ r_2)(4n){:}\Gamma'\{B^\bullet\}^{r_1}\sigma \to t{:}\Gamma'\{A \vee B^\bullet\}^{\sigma'' \circ r'}$$

Set $r := (\sigma'' \circ r') \cup r'' \cup \{(4n, t)\}$ which is a realisation function on the sequent $(\Lambda, [\Gamma'\{A \wedge B^\circ\}]_{4n})$ and by propositional reasoning

$$\mathsf{J} \vdash (\Lambda, [\Gamma'\{A^\circ\}]_{4n})^{r_1}\sigma \to (\Lambda, [\Gamma'\{B^\circ\}]_{4n})^{r_2}\sigma \to (\Lambda, [\Gamma'\{A \wedge B^\circ\}]_{4n})^{r}$$

□

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